A new approach to induction and imprimitivity results

by Stefaan Vaes

Institut de Mathématiques de Jussieu; Algèbres d'Opérateurs et Représentations; 175, rue du Chevaleret; F–75013 Paris (France)
Department of Mathematics; K.U.Leuven; Celestijnenlaan 200B; B-3001 Leuven (Belgium)
e-mail: vaes@math.jussieu.fr

Abstract

In the framework of locally compact quantum groups, we provide an induction procedure for unitary corepresentations as well as coactions on C*-algebras. We prove imprimitivity theorems that unify the existing theorems for actions and coactions of groups. We essentially use von Neumann algebraic techniques.

1 Introduction

The theory of induced representations of locally compact (l.c.) groups was introduced by Mackey [20], who discovered the *imprimitivity theorem*, characterizing induced representations through the presence of a covariant representation of a homogeneous space. Rieffel [25] provided a modern approach using the language of Hilbert C*-modules.

After the work of Rieffel, several induction procedures and imprimitivity results have been obtained, both for actions and for coactions of groups. We shall briefly review them below. The purpose of this paper is to develop such an induction and imprimitivity machinery in the setting of *locally compact quantum groups* [16, 17]. In this way, we provide a unified approach to several results on actions and coactions of groups on C*-algebras. At the same time, our proofs in the setting of l.c. quantum groups are simpler than the classical proofs dealing with coactions of groups.

Another motivation comes from quantum group theory. Meyer and Nest have undertaken a reformulation of the Baum-Connes conjecture [23] in which induction and restriction play a crucial role. The development of induction and imprimitivity in the current paper should play an equally important role in the formulation of a Baum-Connes conjecture for quantum groups.

A final aspect of the paper is the technique that is used to prove the imprimitivity results: we shall obtain Morita equivalences between C*-algebras by using von Neumann algebra techniques and the language of correspondences [5, 24]. This is the main reason why our proofs are simpler than the classical proofs dealing with coactions. Already for group duals, but certainly for l.c. quantum groups, the von Neumann algebra picture of the quantum group (looking at L^∞ rather then C_0) is much more user-friendly.

A first approach to induction of unitary corepresentations of l.c. quantum groups has been developed by Kustermans [14], but without dealing with imprimitivity results or coactions on C*-algebras. Of course, our induction procedure is unitarily equivalent to his. Our approach is simpler and makes it more easy to prove properties (e.g. induction in stages).

In the remainder of the introduction, we shall review several imprimitivity results that were obtained for actions and coactions of groups. We shall explain how they are generalized in the setting of l.c. quantum groups.

Rieffel has given a modern C*-algebraic formulation of Mackey's result using Hilbert C*-modules [25]. The neatest form of Mackey's result is given by the Morita equivalence of C*-algebras

$$G \ {}_f\!\ltimes C_0(G/G_1) \underset{\text{Morita}}{\sim} C^*(G_1) \,, \tag{1.1}$$

whenever G_1 is a closed subgroup of a l.c. group G.

Green [11] has generalized Mackey's induction of unitary representations to C*-dynamical systems. Suppose that G_1 is a closed subgroup of a l.c. group G. Suppose that G_1 acts continuously on a C*-algebra B. Then, Green constructs an induced C*-algebra $\operatorname{Ind} B$ with a continuous action of G such that we obtain a Morita equivalence

$$G \ {}_f\ltimes \operatorname{Ind} B \underset{\text{Morita}}{\sim} G_1 \ {}_f\ltimes B \ . \tag{1.2}$$

If $B = \mathbb{C}$, we get $\operatorname{Ind} B = C_0(G/G_1)$ and we find back Mackey's imprimitivity theorem (1.1).

If a l.c. group G acts continuously on a C*-algebra B and if G_1 is a closed subgroup of G, we can first restrict the action of G to an action of G_1 on B and then induce this restricted action to an action of G. The resulting C*-algebra is $C_0(G/G_1) \otimes B$ with the diagonal action of G. If we now suppose that G_1 is normal in G, Green's imprimitivity theorem can be restated as the Morita equivalence

$$G_1 \ {}_f\ltimes B \underset{\text{Morita}}{\sim} \widehat{G/G_1} \ltimes (G \ {}_f\ltimes B) \ . \tag{1.3}$$

The second crossed product is the crossed product by the restriction of the dual coaction to G/G_1.

Dually, Mansfield proved a coaction version of the Morita equivalence (1.3): if G_1 is a closed normal subgroup of a l.c. group G and if B is a C*-algebra with a *reduced coaction* of G, we have the following Morita equivalence for *reduced crossed products*:

$$\widehat{G/G_1} \ltimes B \underset{\text{Morita}}{\sim} G_1 \ {}_r\ltimes (\widehat{G} \ltimes B) \ . \tag{1.4}$$

In fact, Mansfield had to impose the amenability of G_1 and Kaliszewski & Quigg [13] showed the result for general normal subgroups G_1. The terminology *reduced coaction* is not the standard one: in the literature one uses *normal coaction* (see Definition 2.10).

With representation theory in mind, one wants of course imprimitivity results between full crossed products. Kaliszewski and Quigg have shown recently that (1.4) holds for full crossed products and *maximal coactions* of G (as introduced by Echterhoff, Kaliszewski and Quigg [6]). A maximal coaction is a coaction which is Morita equivalent to a dual coaction on a full crossed product (and a reduced coaction is a coaction which is Morita equivalent to a dual coaction on a reduced crossed product), see Definition 2.13. The same imprimitivity result had been proved before for dual coactions by Echterhoff, Kaliszewski and Raeburn [8].

In Section 6 we define the *quantum homogeneous space*, given a l.c. quantum group and a closed quantum subgroup. We prove a quantum version of the Mackey imprimitivity theorem (1.1). In Section 7, we study dynamical systems. Given a coaction of a closed quantum subgroup (A_1, Δ_1) of a l.c. quantum group (A, Δ) on a C*-algebra B, we construct an induced C*-algebra $\operatorname{Ind} B$ with a coaction of (A, Δ) such that a quantum version of (1.2) holds. Observe that already for coactions of groups such a construction of induced coactions was not known up to now.

In Section 8 we describe what happens if we first restrict and then induce a coaction. Instead of a tensor product, we obtain a twisted product of the original C*-algebra and the quantum homogeneous space with some kind of diagonal coaction. This is used in Section 10 to obtain a quantum version of (1.4), both for reduced crossed products (and reduced coactions) and for full crossed products (and maximal coactions). In fact we define crossed products by homogeneous spaces and hence, we do not have to assume normality of the quantum subgroup.

In [7], Echterhoff, Kaliszewski, Quigg and Raeburn discuss naturality of the imprimitivity theorems for actions and coactions of groups. In our general approach, we also get covariant Morita equivalences and naturality.

As stated above, the technique used in this paper is von Neumann algebraic in nature. After a section of preliminaries on locally compact quantum groups, closed quantum subgroups and crossed products, we provide a von Neumann algebraic approach to representation theory for quantum groups in Section 3. This is used to give an easy approach to induction of representations in Section 4, simplifying the original approach by Kustermans [14]. In Section 5 we prove a preliminary imprimitivity theorem, which is the crucial ingredient in the next sections that we already discussed above.

Notation 1.1. The most over-used symbol of this paper is $\otimes$. It shall be used to denote tensor products of Hilbert spaces and von Neumann algebras, as well as *minimal* tensor products of C*-algebras.

The multiplier algebra of a C*-algebra A is denoted by $\mathcal{M}(A)$.

When X is a subset of a Banach space, we denote by $[X]$ the closed linear span of X.

2 Preliminaries

2.1 Locally compact quantum groups

We use [16, 17] as references for the theory of locally compact (l.c.) quantum groups. Since the definition of a l.c. quantum group in [16, 17] is based on the existence of invariant weights (Haar measures), we introduce the following weight theoretic notation.

Let φ be a normal, semi-finite, faithful (n.s.f.) weight on a von Neumann algebra M. Then, we write

$$\mathcal{M}_\varphi^+ = \{x \in M^+ \mid \varphi(x) < \infty\} \quad \text{and} \quad \mathcal{N}_\varphi = \{x \in M \mid \varphi(x^*x) < \infty\} \ .$$

Definition 2.1. A pair (M, Δ) is called a (von Neumann algebraic) l.c. quantum group when

- M is a von Neumann algebra and $\Delta : M \to M \otimes M$ is a normal and unital *-homomorphism satisfying the coassociativity relation : $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$;

- there exist n.s.f. weights φ and ψ on M such that

  - φ is left invariant in the sense that $\varphi\big((\omega \otimes \iota)\Delta(x)\big) = \varphi(x)\omega(1)$ for all $x \in \mathcal{M}_\varphi^+$ and $\omega \in M_*^+$,
  - ψ is right invariant in the sense that $\psi\big((\iota \otimes \omega)\Delta(x)\big) = \psi(x)\omega(1)$ for all $x \in \mathcal{M}_\psi^+$ and $\omega \in M_*^+$.

Fix such a l.c. quantum group (M, Δ).

Represent M in the GNS-construction of φ with GNS-map $\Lambda : \mathcal{N}_\varphi \to H$. We define a unitary W on $H \otimes H$ by

$$W^*(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)(\Delta(b)(a \otimes 1)) \quad \text{for all } a, b \in \mathcal{N}_\varphi \ .$$

Here, $\Lambda \otimes \Lambda$ denotes the canonical GNS-map for the tensor product weight $\varphi \otimes \varphi$. One proves that W satisfies the pentagonal equation: $W_{12}W_{13}W_{23} = W_{23}W_{12}$, and we say that W is a *multiplicative unitary*. It is the *left regular corepresentation*. The von Neumann algebra M is the strong closure of the algebra $\{(\iota \otimes \omega)(W) \mid \omega \in B(H)_*\}$ and $\Delta(x) = W^*(1 \otimes x)W$, for all $x \in M$. Next, the l.c. quantum group (M, Δ) has an antipode S, which is the unique σ-strong* closed linear map from M to M satisfying $(\iota \otimes \omega)(W) \in D(S)$ for all $\omega \in B(H)_*$ and $S(\iota \otimes \omega)(W) = (\iota \otimes \omega)(W^*)$ and such that the elements $(\iota \otimes \omega)(W)$ form a σ-strong* core for S. The antipode S has a polar decomposition $S = R\tau_{-i/2}$ where R is an anti-automorphism of M and (τ_t) is a strongly continuous one-parameter group of automorphisms of M. We call R the *unitary antipode* and (τ_t) the *scaling group* of (M, Δ). From [16], Proposition 5.26 we know that $\sigma(R \otimes R)\Delta = \Delta R$. So φR is a right invariant weight on (M, Δ) and we take $\psi := \varphi R$.

The *dual l.c. quantum group* $(\hat{M}, \hat{\Delta})$ is defined in [16], Section 8. Its von Neumann algebra $\hat{M}$ is the strong closure of the algebra $\{(\omega \otimes \iota)(W) \mid \omega \in B(H)_*\}$ and the comultiplication is given by $\hat{\Delta}(x) = \Sigma W(x \otimes 1)W^*\Sigma$ for all $x \in \hat{M}$. On $\hat{M}$ there exists a canonical left invariant weight $\hat{\varphi}$ and the associated multiplicative unitary is denoted by $\hat{W}$. From [16], Proposition 8.16, it follows that $\hat{W} = \Sigma W^*\Sigma$.

Since $(\hat{M}, \hat{\Delta})$ is again a l.c. quantum group, we can introduce the antipode $\hat{S}$, the unitary antipode $\hat{R}$ and the scaling group $(\hat{\tau}_t)$ exactly as we did it for (M, Δ).

We shall denote the modular conjugations of the weights φ and $\hat{\varphi}$ by J and $\hat{J}$ respectively. The operators J and $\hat{J}$ are anti-unitary involutions on the Hilbert space H. They implement the unitary antipodes in the sense that

$$R(x) = \hat{J}x^*\hat{J} \quad \text{for all } x \in M \qquad \text{and} \qquad \hat{R}(y) = Jy^*J \quad \text{for all } y \in \hat{M} \ .$$

From modular theory, we also know that $M' = JMJ$ and $\hat{M}' = \hat{J}\hat{M}\hat{J}$.

We already discussed the left regular corepresentations W and $\hat{W}$ of (M, Δ) and $(\hat{M}, \hat{\Delta})$, respectively. These are multiplicative unitaries and $\hat{W} = \Sigma W^* \Sigma$. We observe moreover that $W \in M \otimes \hat{M}$ and $\hat{W} \in \hat{M} \otimes M$. Since we also have right invariant weights on (M, Δ) and $(\hat{M}, \hat{\Delta})$, we consider as well the right regular corepresentations $V \in \hat{M}' \otimes M$ and $\hat{V} \in M' \otimes \hat{M}$. These are also multiplicative unitaries and satisfy

$$V = (\hat{J} \otimes \hat{J})\hat{W}(\hat{J} \otimes \hat{J}) , \quad \hat{V} = (J \otimes J)W(J \otimes J) .$$

We finally mention the formula $W^* = (\hat{J} \otimes J)W(\hat{J} \otimes J)$, which is equivalent to saying that $(R \otimes \hat{R})(W) = W$.

Every l.c. quantum group has an *associated C*-algebra* of 'continuous functions tending to zero at infinity' and we denote it by A (resp. $\hat{A}$):

$$A := [(\iota \otimes \omega)(W) \mid \omega \in \mathrm{B}(H)_*] , \quad \hat{A} = [(\omega \otimes \iota)(W) \mid \omega \in \mathrm{B}(H)_*] .$$

It is clear that $A \subset M$. Also, the comultiplication Δ restricts to A and yields a comultiplication $\Delta : A \rightarrow \mathcal{M}(A \otimes A)$.

Notation 2.2. When we shall be dealing with coactions of l.c. quantum groups on C*-algebras, we will make use all the time of the C*-algebraic picture (A, Δ) of our l.c. quantum group. So, we shall speak about *the l.c. quantum group* (A, Δ).

2.2 Closed quantum subgroups

We first discuss the notion of a *morphism* between l.c. quantum groups: see the work of Kustermans [15] for details. We explained that every l.c. quantum groups admits a C*-algebra A and a dual C*-algebra $\hat{A}$. In the classical case of l.c. groups, this comes down to the C*-algebras $C_0(G)$ and $C_r^*(G)$. But there is of course as well the universal C*-algebra $C^*(G)$.

Definition 2.3. Let (A, Δ) be a l.c. quantum group. A *unitary corepresentation* of (A, Δ) on a C*-B-module $\mathcal{E}$ is a unitary $X \in \mathcal{L}(A \otimes \mathcal{E})$ satisfying

$$(\Delta \otimes \iota)(X) = X_{13}X_{23} .$$

In [15] the universal dual $(\hat{A}^{\mathrm{u}}, \hat{\Delta}^{\mathrm{u}})$ of (A, Δ) is defined. This definition is analogous to the definition of the full group C*-algebra $C^*(G)$ of a l.c. group G. This means that there exists a *universal corepresentation* $\mathcal{W} \in \mathcal{M}(A \otimes \hat{A}^{\mathrm{u}})$ such that the formula

$$(\iota \otimes \theta)(\mathcal{W}) = X$$

gives a bijective correspondence between non-degenerate *-homomorphisms $\theta : \hat{A}^{\mathrm{u}} \rightarrow \mathcal{L}(\mathcal{E})$ and unitary corepresentations $X \in \mathcal{L}(A \otimes \mathcal{E})$, whenever $\mathcal{E}$ is a C*-B-module. Moreover, the comultiplication $\hat{\Delta}^{\mathrm{u}}$ satisfies

$$(\iota \otimes \hat{\Delta}^{\mathrm{u}})(\mathcal{W}) = \mathcal{W}_{13}\mathcal{W}_{12} .$$

In exactly the same way, there is a universal version of (A, Δ), which is denoted by $(A^{\mathrm{u}}, \Delta^{\mathrm{u}})$. There exists a universal corepresentation $\hat{\mathcal{W}} \in \mathcal{M}(A^{\mathrm{u}} \otimes \hat{A})$ of $(\hat{A}, \hat{\Delta})$ such that the formula $(\theta \otimes \iota)(\hat{\mathcal{W}}) = X$ gives a bijective correspondence between non-degenerate *-homomorphisms $\theta : A^{\mathrm{u}} \rightarrow \mathcal{L}(\mathcal{E})$ and unitary corepresentations $X \in \mathcal{L}(\mathcal{E} \otimes \hat{A})$, whenever $\mathcal{E}$ is a C*-B-module.

Definition 2.4. A morphism $(M, \Delta) \xrightarrow{\pi} (M_1, \Delta_1)$ between the l.c. quantum groups (M, Δ) and (M_1, Δ_1) is a non-degenerate *-homomorphism

$$\pi : A^{\mathrm{u}} \rightarrow \mathcal{M}(A_1^{\mathrm{u}}) \quad \text{satisfying} \quad \Delta_1^{\mathrm{u}}\pi = (\pi \otimes \pi)\Delta^{\mathrm{u}} .$$

We slightly abuse notation by writing $(M, \Delta) \xrightarrow{\pi} (M_1, \Delta_1)$ and we should always keep in mind that π lives on the level of universal C*-algebras.

Observe that in the classical situation, this corresponds to $A^{\mathrm{u}} = C_0(G)$, $A_1^{\mathrm{u}} = C_0(G_1)$ and $\pi(f) = f \circ \theta$, where $\theta : G_1 \to G$ is a continuous group homomorphism. Associated with θ, we can then write as well $\hat{\pi} : C^*(G_1) \to \mathcal{M}(C^*(G))$ defined by $\hat{\pi}(\lambda_p) = \lambda_{\theta(p)}$. In the same way, every morphism $(M, \Delta) \xrightarrow{\pi} (M_1, \Delta_1)$ between l.c. quantum groups admits canonically a dual morphism $(\hat{M}_1, \hat{\Delta}_1) \xrightarrow{\hat{\pi}} (\hat{M}, \hat{\Delta})$.

Definition 2.5. We say that the morphism $(M, \Delta) \xrightarrow{\pi} (M_1, \Delta_1)$ identifies (M_1, Δ_1) as a *closed quantum subgroup* of (M, Δ) if there exists a faithful, normal, unital *-homomorphism $\hat{M}_1 \to \hat{M}$ which makes the following diagram commute.

$$
\begin{array}{ccc}
\hat{A}_1^{\mathrm{u}} & \xrightarrow{\hat{\pi}} & \mathcal{M}(\hat{A}^{\mathrm{u}}) \\
\downarrow & & \downarrow \\
\hat{M}_1 & \longrightarrow & \hat{M}
\end{array}
$$

In that case, we continue writing $\hat{\pi} : \hat{M}_1 \to \hat{M}$.

One verifies that, in the classical setting where $\pi(f) = f \circ \theta$ for a continuous group homomorphism $\theta : G_1 \to G$, this comes down to the fact that θ identifies G_1 with a closed subgroup of G. Indeed, the map $\hat{\pi}(\lambda_p) = \lambda_{\theta(p)}$ extends to a faithful, normal *-homomorphism $\mathcal{L}(G_1) \to \mathcal{L}(G)$ if and only if G_1 is a closed subgroup of G.

2.3 Crossed products and regularity

Definition 2.6. A *coaction* of a l.c. quantum group (A, Δ) on a C*-algebra B is a non-degenerate *-homomorphism

$$\alpha : B \to \mathcal{M}(A \otimes B) \quad \text{satisfying} \quad (\iota \otimes \alpha)\alpha = (\Delta \otimes \iota)\alpha .$$

We say that α is a *continuous coaction* if

$$[\alpha(B)(A \otimes 1)] = A \otimes B .$$

Coactions and their associated crossed products have been studied in detail by Baaj and Skandalis [1]. We recall some basic concepts. In this paper, we shall make use as well of coactions on C*-modules. This is discussed in detail in the appendix, following another paper of Baaj and Skandalis [2].

Let $\alpha : B \to \mathcal{M}(A \otimes B)$ be a continuous coaction of a l.c. quantum group (A, Δ) on a C*-algebra B. Then,

$$[\alpha(B)(\hat{A} \otimes 1)] \subset \mathcal{M}(\mathcal{K}(H) \otimes B)$$

is a C*-algebra which is called the *reduced crossed product* and denoted by $\hat{A} \ _r\ltimes B$.

A pair (X, π) consisting of a unitary corepresentation $X \in \mathcal{M}(A \otimes \mathcal{K}(K))$ of (A, Δ) on a Hilbert space K and a non-degenerate *-homomorphism $\pi : B \to \mathrm{B}(K)$ is called a *covariant representation* of α if

$$(\iota \otimes \pi)\alpha(x) = X^*(1 \otimes \pi(x))X \quad \text{for all} \quad x \in B .$$

There exists a unique (up to isomorphism) C*-algebra denoted by $\hat{A}^{\mathrm{u}} \ _f\ltimes B$ and called the *full crossed product*, equipped with a universal covariant representation

$$X_{\mathrm{u}} \in \mathcal{M}(A \otimes (\hat{A}^{\mathrm{u}} \ _f\ltimes B)) , \quad \pi_{\mathrm{u}} : B \to \mathcal{M}(\hat{A}^{\mathrm{u}} \ _f\ltimes B)$$

such that the formulas

$$X = (\iota \otimes \theta)(X_{\mathrm{u}}) \quad \text{and} \quad \pi = \theta\pi_{\mathrm{u}}$$

yield a bijective correspondence between covariant representations (X, π) of α and non-degenerate representations θ of the C*-algebra $\hat{A}^{\mathrm{u}} \ _f\ltimes B$.

By definition it is clear that $\hat{A}^{\mathrm{u}}$ coincides with the full crossed product of (A, Δ) coacting on the trivial C*-algebra $\mathbb{C}$. It is also clear that there is a natural surjective *-homomorphism $\hat{A}^{\mathrm{u}} \ _f\ltimes B \to \hat{A} \ _r\ltimes B$.

Remark 2.7. From time to time we shall use as well right coactions $\alpha : B \to \mathcal{M}(B \otimes A)$, satisfying $(\alpha \otimes \iota)\alpha = (\iota \otimes \Delta)\alpha$. The reduced crossed product is then given by

$$B \rtimes_r \hat{A}^{\mathrm{op}} = [\alpha(B)(1 \otimes \hat{J}\hat{A}\hat{J})] .$$

The reason why the opposite algebra $\hat{J}\hat{A}\hat{J}$ appears is natural: a right coaction of (A, Δ) corresponds to a left coaction of $(A, \Delta^{\mathrm{op}})$. The dual of the opposite quantum group $(A, \Delta^{\mathrm{op}})$ is $\hat{J}\hat{A}\hat{J}$. In the same way, one defines $B \rtimes_f \hat{A}^{\mathrm{u,op}}$.

Both the reduced and the full crossed products admit a *dual coaction* of $(\hat{A}, \hat{\Delta}^{\mathrm{op}})$, which leaves invariant B and acts as the comultiplication $\hat{\Delta}^{\mathrm{op}}$ on $\hat{A}$. So, it is a natural question to consider what happens with the second crossed products

$$A^{\mathrm{op}} {}_r\ltimes (\hat{A} {}_r\ltimes B) \quad \text{and} \quad A^{\mathrm{u,op}} {}_f\ltimes (\hat{A}^{\mathrm{u}} {}_f\ltimes B) .$$

For abelian l.c. groups, it is well known that $\hat{G} \ltimes G \ltimes B \cong \mathcal{K}(L^2(G)) \otimes B$. This result need no longer be true for l.c. quantum groups. Indeed, taking $B = \mathbb{C}$, it might very well be the case that $A^{\mathrm{op}} {}_r\ltimes \hat{A} \ncong \mathcal{K}(H)$. The following definition, due to Baaj and Skandalis [1], describes the quantum groups for which the classical biduality result holds.

For any multiplicative unitary W on a Hilbert space H, we introduce (following [1]) the algebra $\mathcal{C}(W)$ by the formula

$$\mathcal{C}(W) := \{(\iota \otimes \omega)(\Sigma W) \mid \omega \in \mathrm{B}(H)_*\} .$$

Definition 2.8. A l.c. quantum group (A, Δ) is said to be *regular* if $[\mathcal{C}(W)] = \mathcal{K}(H)$, where W denotes the left regular representation.

It follows from Proposition 2.6 in [3] that a l.c. quantum group is regular if and only if the reduced crossed product of A and $\hat{A}$ is isomorphic with $\mathcal{K}(H)$.

Remark 2.9. The notion of *continuous coaction* is somehow problematic for non-regular quantum groups. Definition 2.6 makes sense, but is not the only natural definition in the non-regular case. See [3] for a detailed discussion.

Definition 2.10. A continuous coaction $\alpha : B \to \mathcal{M}(A \otimes B)$ of (A, Δ) on B is said to be *reduced* if α is a faithful *-homomorphism.

Whenever α is a reduced continuous coaction of a regular l.c. quantum group (A, Δ) on a C*-algebra B, we have that

$$A^{\mathrm{op}} {}_r\ltimes (\hat{A} {}_r\ltimes B) \cong \mathcal{K}(H) \otimes B .$$

In fact, one obtains a natural covariant Morita equivalence $A^{\mathrm{op}} {}_r\ltimes (\hat{A} {}_r\ltimes B) \underset{\text{Morita}}{\sim} B$ with respect to the bidual coaction on the double crossed product and the coaction α on B.

All dual coactions on reduced crossed products are reduced coactions and the biduality theorem shows that, in fact, a continuous coaction is reduced if and only if it is Morita equivalent to a dual coaction.

Definition 2.11. A l.c. quantum group (A, Δ) is said to be *strongly regular* if $\hat{A}^{\mathrm{u}} {}_f\ltimes A \cong \mathcal{K}(H)$.

Since we always have the surjective *-homomorphism $\pi : \hat{A}^{\mathrm{u}} {}_f\ltimes A \to \hat{A} {}_r\ltimes A$, it follows that a l.c. quantum group is strongly regular if and only if it is regular and the *-homomorphism π is faithful.

Remark 2.12. Not every l.c. quantum group is regular. Non-regular examples are given by the quantum groups $E_\mu(2)$, $ax + b$ or certain bicrossed products (see [3] for a detailed discussion of the latter case). Examples of regular quantum groups include all the compact or discrete quantum groups, all Kac algebras and a wide class of bicrossed products. Also the analytic versions of the algebraic quantum groups [18] are regular.

All the above mentioned examples of regular quantum groups are in fact strongly regular. It is not known whether there exist regular quantum groups which are not strongly regular.

We choose to define coactions as *-homomorphisms $\alpha : B \to \mathcal{M}(A \otimes B)$, where A is the *reduced* C*-algebra of the l.c. quantum group. Of course, we can define a continuous coaction of the universal quantum group $(A^{\mathrm{u}}, \Delta^{\mathrm{u}})$ on the C*-algebra B as a non-degenerate *-homomorphism $\alpha : B \to \mathcal{M}(A^{\mathrm{u}} \otimes B)$ satisfying $(\iota \otimes \alpha)\alpha = (\Delta^{\mathrm{u}} \otimes \iota)\alpha$ and $[\alpha(B)(A^{\mathrm{u}} \otimes 1)] = A^{\mathrm{u}} \otimes B$. Observe that it follows automatically that $(\varepsilon \otimes \iota)\alpha(x) = x$ for all $x \in B$, where $\varepsilon : A^{\mathrm{u}} \to \mathbb{C}$ denotes the co-unit of $(A^{\mathrm{u}}, \Delta^{\mathrm{u}})$. If we denote by $\pi : A^{\mathrm{u}} \to A$ the natural surjective *-homomorphism, it is clear that $(\pi \otimes \iota)\alpha$ will be a continuous coaction of (A, Δ) whenever α is a continuous coaction of $(A^{\mathrm{u}}, \Delta^{\mathrm{u}})$. We briefly discuss the converse: when does continuous coactions of (A, Δ) lift to continuous coactions of $(A^{\mathrm{u}}, \Delta^{\mathrm{u}})$?

Fischer has shown ([10], Proposition 3.26) that a *reduced* continuous coaction has a unique lift to a continuous coaction of $(A^{\mathrm{u}}, \Delta^{\mathrm{u}})$. His proof works for general l.c. quantum groups. For regular quantum groups, this is obvious: a reduced coaction is Morita equivalent to a dual coaction on a reduced crossed product and it is clear that dual coactions admit a lift.

Another class of coactions for which such a unique lift exists are the so-called maximal coactions introduced in [10, 6].

Definition 2.13. Let (A, Δ) be a *regular* l.c. quantum group. A continuous coaction $\alpha : B \to \mathcal{M}(A \otimes B)$ is said to be *maximal* if the natural surjective *-homomorphism

$$A^{\mathrm{u,op}} {}_f\ltimes \hat{A}^{\mathrm{u}} {}_f\ltimes B \to \mathcal{K}(H) \otimes B$$

is an isomorphism.

Almost by definition, a maximal coaction is a coaction which is Morita equivalent to a dual coaction on a full crossed product. It is then clear that maximal coactions admit a unique lift to the universal level.

Up to now we discussed coactions on C*-algebras. Of course, quantum groups can coact as well on von Neumann algebras. We refer to [26] for details.

Definition 2.14. A coaction of a l.c. quantum group (M, Δ) on a von Neumann algebra N is a faithful, normal, unital *-homomorphism $\alpha : N \to M \otimes N$ satisfying $(\iota \otimes \alpha)\alpha = (\Delta \otimes \iota)\alpha$.

The *crossed product* $\hat{M} \ltimes N$ is the von Neumann subalgebra of $\mathrm{B}(H) \otimes N$ generated by $\alpha(N)$ and $\hat{M} \otimes 1$.

Again the crossed product $\hat{M} \ltimes N$ carries a *dual coaction* of $(\hat{M}, \hat{\Delta}^{\mathrm{op}})$ and the following biduality result holds: $M' \ltimes \hat{M} \ltimes N \cong \mathrm{B}(H) \otimes N$.

3 Four different pictures of corepresentation theory

In the preliminary section, we defined unitary corepresentations of l.c. quantum groups and discussed the bijective correspondence with non-degenerate representations of the universal dual quantum group. This yields two different pictures of corepresentation theory. In this section we present two other useful pictures, which are von Neumann algebraic. These pictures are a major tool in the rest of the paper.

3.1 Bicovariant C*-correspondences

Before we present the third picture of unitary corepresentation theory, we give the following definition generalizing the notion of a correspondence [5, 24, 4] from Hilbert spaces to C*-modules. We shall need to represent *von Neumann algebras* on C*-modules.

Definition 3.1. Let N be a von Neumann algebra and $\mathcal{E}$ a C*-B-module. A unital *-homomorphism $\pi : N \to \mathcal{L}(\mathcal{E})$ is said to be *strict* (or *normal*) if it is strong* continuous on the unit ball of N.

Recall that the strong* topology on the unit ball of $\mathcal{L}(\mathcal{E})$ coincides with the strict topology identifying $\mathcal{L}(\mathcal{E}) = \mathcal{M}(\mathcal{K}(\mathcal{E}))$. That motivates the terminology of a strict *-homomorphism.

The following is the almost standard example of a strict *-homomorphism. Suppose that $N \subset \mathrm{B}(H)$ is a von Neumann algebra and consider the C*-B-module $H \otimes B$. Let $V \in \mathcal{L}(H \otimes B) = \mathcal{M}(\mathcal{K}(H) \otimes B)$ be a unitary operator. Then,

$$N \to \mathcal{L}(H \otimes B) : x \mapsto V(x \otimes 1)V^*$$

is a strict *-homomorphism.

Definition 3.2. Let M and N be von Neumann algebras. We say that a C*-B-module $\mathcal{E}$ is a B-correspondence from N to M if we have

- a strict *-homomorphism $\pi_\ell : M \to \mathcal{L}(\mathcal{E})$,

- a strict *-antihomomorphism $\pi_r : N \to \mathcal{L}(\mathcal{E})$,

such that $\pi_\ell(M)$ and $\pi_r(N)$ commute.

Notation 3.3. A B-correspondence from N to M will be denoted as $_M\boxed{\mathcal{E}}_N$. We will write the left and right module actions as

$$x \cdot v = \pi_\ell(x)v \quad \text{and} \quad v \cdot y = \pi_r(y)v \quad \text{for all } x \in M, y \in N, v \in \mathcal{E} \ .$$

From [5], we know how to construct a correspondence from the group von Neumann algebra $\mathcal{L}(G)$ to $\mathcal{L}(G)$ given a unitary representation of G. We can do the same thing for l.c. quantum groups.

Proposition 3.4. *Let (A, Δ) be a l.c. quantum group and $X \in \mathcal{L}(A \otimes \mathcal{E})$ a unitary corepresentation on a C*-B-module $\mathcal{E}$.*

Then, there is a B-correspondence $_{\hat{M}}\boxed{H \otimes \mathcal{E}}_{\hat{M}}$ given by

$$x \cdot v = X(x \otimes 1)X^*v \quad \text{and} \quad v \cdot y = (\hat{J}y^*\hat{J} \otimes 1)v \quad \text{for } x, y \in \hat{M}, \ v \in H \otimes \mathcal{E} \ .$$

To prove this proposition, we only have to observe that $(\iota \otimes \pi_\ell)(W) = W_{12}X_{13}$, where $\pi_\ell : \hat{M} \to \mathcal{L}(\mathcal{E})$ denotes the left module action $\pi_\ell(x)v = x \cdot v$. Hence, the left and right module actions commute.

We now want to characterize which B-correspondences from $\hat{M}$ to $\hat{M}$ come from a unitary corepresentation. Let $X \in \mathcal{L}(A \otimes \mathcal{E})$ be a unitary corepresentation of (A, Δ) on a C*-B-module $\mathcal{E}$ and make the B-correspondence $_{\hat{M}}\boxed{H \otimes \mathcal{E}}_{\hat{M}}$. Then, we also have a strict *-homomorphism $\pi : M' \to \mathcal{L}(H \otimes \mathcal{E}) : \pi(x) = x \otimes 1$ which is covariant with respect to both the left and right module action of $\hat{M}$, in the following precise sense.

Definition 3.5. Let $_{\hat{M}}\boxed{\mathcal{F}}_{\hat{M}}$ be a B-correspondence from $\hat{M}$ to $\hat{M}$ and suppose that $\pi : M' \to \mathcal{L}(\mathcal{F})$ is a strict *-homomorphism. We say that π is *bicovariant* when

$$(\pi_\ell \otimes \iota)\hat{\Delta}(x) = (\pi \otimes \iota)(\hat{V})(\pi_\ell(x) \otimes 1)(\pi \otimes \iota)(\hat{V}^*) \quad \text{and} \quad (\pi_r \otimes \hat{R})\hat{\Delta}(x) = (\pi \otimes \iota)(\hat{V}^*)(\pi_r(x) \otimes 1)(\pi \otimes \iota)(\hat{V})$$

for all $x \in \hat{M}$. Here π_ℓ and π_r denote the left and right module actions of $\hat{M}$ on $\mathcal{F}$.

In that case, we call $\mathcal{F}$ a *bicovariant B-correspondence* and we write $_{\hat{M}}\boxed{\mathcal{F}}^{M'}_{\hat{M}}$.

Remark 3.6. Let $_{\hat{M}}\boxed{\mathcal{F}}_{\hat{M}}$ be a B-correspondence from $\hat{M}$ to $\hat{M}$. Let $Y \in \mathcal{L}(\mathcal{F} \otimes \hat{A})$ be a unitary corepresentation of $(\hat{A}, \hat{\Delta})$, which means that $(\iota \otimes \hat{\Delta})(Y) = Y_{12}Y_{13}$. Suppose that Y is bicovariant in the sense that

$$(\pi_\ell \otimes \iota)\hat{\Delta}(x) = Y(\pi_\ell(x) \otimes 1)Y^* \quad \text{and} \quad (\pi_r \otimes \hat{R})\hat{\Delta}(x) = Y^*(\pi_r(x) \otimes 1)Y \tag{3.1}$$

for all $x \in \hat{M}$.

Then, there exists a unique strict *-homomorphism $\pi : M' \to \mathcal{L}(\mathcal{F})$ which is bicovariant and satisfies $Y = (\pi \otimes \iota)(\hat{V})$.

To show this, you only need $\pi_\ell : \hat{M} \to \mathcal{L}(\mathcal{F})$ satisfying $(\pi_\ell \otimes \iota)\hat{\Delta}(x) = Y(\pi_\ell(x) \otimes 1)Y^*$. Writing $X = (\iota \otimes \pi_\ell)(\hat{V})$, this means that $X_{12}\hat{V}_{13} = Y_{23}X_{12}Y_{23}^*$.

On the other hand, Y is a corepresentation, which means that $W_{23}Y_{12}W_{23}^* = Y_{13}Y_{12}$. Recall now that $\hat{V} = (J\hat{J} \otimes 1)W^*(\hat{J}J \otimes 1)$. So, if we define $\hat{Y} = (J\hat{J} \otimes 1)\Sigma Y \Sigma(\hat{J}J \otimes 1) \in \mathcal{L}(H \otimes \mathcal{F})$, we get that $\hat{V}_{13}^*\hat{Y}_{12}\hat{V}_{13} = Y_{23}\hat{Y}_{12}$. Together with the formula in the previous paragraph, we find that

$$(X\hat{Y})_{12}\hat{V}_{13}(X\hat{Y})_{12}^* = Y_{23} .$$

From this, it follows that there exists a strict *-homomorphism $\pi : M' \to \mathcal{L}(\mathcal{F})$ such that

$$(X\hat{Y})(x \otimes 1)(X\hat{Y})^* = 1 \otimes \pi(x)$$

for all $x \in M'$. Then also $Y = (\pi \otimes \iota)(\hat{V})$.

So, a bicovariant B-correspondence is determined by a B-correspondence $\mathcal{F}$ between $\hat{M}$ and $\hat{M}$ together with a corepresentation $Y \in \mathcal{L}(\mathcal{F} \otimes \hat{A})$ satisfying the bicovariance relations (3.1).

The following proposition provides the third equivalent picture of corepresentation theory as *the theory of bicovariant B-correspondences*.

Proposition 3.7. *If* ${}_{\hat{M}}\boxed{\overset{M'}{\mathcal{F}}}{}_{\hat{M}}$ *is a bicovariant B-correspondence, there exists a canonically determined C^*-B-module $\mathcal{E}$ and a corepresentation $X \in \mathcal{L}(A \otimes \mathcal{E})$ such that*

$$ {}_{\hat{M}}\boxed{\overset{M'}{\mathcal{F}}}{}_{\hat{M}} \quad \simeq \quad {}_{\hat{M}}\boxed{\overset{M'}{H \otimes \mathcal{E}}}{}_{\hat{M}} $$

as bicovariant correspondences. So, we get a bijective relation between unitary corepresentations on C^*-B-modules and bicovariant B-correspondences.

Proof. Suppose that ${}_{\hat{M}}\boxed{\overset{M'}{\mathcal{F}}}{}_{\hat{M}}$ is a bicovariant B-correspondence. Using the technique of the proof of Lemma 5.5 in [3], we get a strict *-homomorphism $\theta : \mathrm{B}(H) \to \mathcal{L}(\mathcal{F})$ such that $\pi(x) = \theta(x)$ for all $x \in M'$ and $\pi_r(y) = \theta(\hat{J}y^*\hat{J})$ for all $y \in \hat{M}$. Since M' and $\hat{M}'$ generate $\mathrm{B}(H)$ as a von Neumann algebra, the strict *-homomorphism θ is canonically defined.

Using $\theta : \mathrm{B}(H) \to \mathcal{L}(\mathcal{F})$, we get a canonical C^*-B-module $\mathcal{E}$ and an isomorphism $\mathcal{F} \simeq H \otimes \mathcal{E}$ such that $\theta(x)$ becomes $x \otimes 1$ under this isomorphism. One can, for instance, define $\mathcal{E}$ to be the space of bounded linear maps $v : H \to \mathcal{F}$ satisfying $vx = \theta(x)v$ for all $x \in \mathrm{B}(H)$.

The isomorphism $\mathcal{F} \simeq H \otimes \mathcal{E}$ yields $\pi_\ell : \hat{M} \to \mathcal{L}(H \otimes \mathcal{E})$ such that the range of π_ℓ commutes with $\hat{M}' \otimes 1$ and such that $(\pi_\ell \otimes \iota)\hat{\Delta}(x) = \hat{V}_{13}(\pi_\ell(x) \otimes 1)\hat{V}_{13}^*$.

Write $Z = W_{12}^*(\iota \otimes \pi_\ell)(W) \in \mathcal{L}(A \otimes H \otimes \mathcal{E})$. Then, Z commutes with $1 \otimes \hat{M}' \otimes 1$. On the other hand,

$$\hat{V}_{24}Z_{123}\hat{V}_{24}^* = (\iota \otimes \hat{\Delta})(W^*)_{124}(\iota \otimes \pi_\ell \otimes \iota)(\iota \otimes \hat{\Delta})(W) = W_{12}^*W_{14}^*W_{14}(\iota \otimes \pi_\ell)(W)_{123} = Z_{123} .$$

Hence, Z commutes with $1 \otimes M' \otimes 1$. This implies that there exists $X \in \mathcal{L}(A \otimes \mathcal{E})$ such that $Z = X_{13}$. It follows that

$$(\iota \otimes \pi_\ell)(W) = W_{12}X_{13} .$$

From this, we conclude that X is a unitary corepresentation of (A, Δ) in $\mathcal{E}$ and that

$$ {}_{\hat{M}}\boxed{\overset{M'}{\mathcal{F}}}{}_{\hat{M}} \quad \simeq \quad {}_{\hat{M}}\boxed{\overset{M'}{H \otimes \mathcal{E}}}{}_{\hat{M}} $$

as bicovariant correspondences. $\qquad\square$

3.2 Bicovariant W*-bimodules

We present a fourth picture of corepresentation theory, which only works to describe corepresentations on Hilbert spaces rather then C*-modules.

Definition 3.8. Let M, N be von Neumann algebras. A W*-M-N-bimodule $\mathcal{E}$ is a W*-N-module equipped with a normal, unital *-homomorphism $\pi_\ell : M \to \mathcal{L}(\mathcal{E})$.

Proposition 3.9. *Let $X \in M \otimes \mathrm{B}(K)$ be a unitary corepresentation of a l.c. quantum group (M, Δ) on a Hilbert space K. Consider the W*-$\hat{M}$-module $\hat{M} \otimes K$. Then, there exists a unique normal, unital *-homomorphism $\pi_\ell : \hat{M} \to \mathcal{L}(\hat{M} \otimes K)$ satisfying $(\iota \otimes \pi_\ell)(W) = W_{12} X_{13}$.*

As such, $\hat{M} \otimes K$ becomes a W-$\hat{M}$-$\hat{M}$-bimodule.*

Proof. We define $\pi_\ell : \hat{M} \to \mathrm{B}(H \otimes K) : \pi_\ell(x) = X(x \otimes 1)X^*$. Then, $(\iota \otimes \pi_\ell)(W) = W_{12} X_{13}$. Hence, $\pi_\ell(\hat{M}) \subset \hat{M} \otimes K = \mathcal{L}(\hat{M} \otimes K)$. So, we are done. $\qquad\square$

Exactly as we characterized the C*-correspondences coming from a corepresentation, we now characterize the W*-bimodules coming from a corepresentation. In order to do so, we make use of coactions on W*-modules. We refer to the appendix for details on this topic.

If $X \in M \otimes \mathrm{B}(K)$ is a unitary corepresentation of (M, Δ) on a Hilbert space K, we construct the coaction

$$\gamma : \hat{M} \otimes K \to (\hat{M} \otimes K) \otimes \hat{M} : \gamma(z) = (\hat{\Delta} \otimes \iota)(z)_{132}$$

on the W*-$\hat{M}$-module $\hat{M} \otimes K$ which is compatible with the right coaction $\hat{\Delta}$ on $\hat{M}$. Moreover, we observe that

$$\gamma \pi_\ell = (\pi_\ell \otimes \iota)\hat{\Delta} \ ,$$

where we still write γ for the associated coaction on $\mathcal{L}(\hat{M} \otimes K)$.

This leads to the following definition.

Definition 3.10. Let (M, Δ) be a l.c. quantum group and let $\mathcal{F}$ be a W*-$\hat{M}$-$\hat{M}$-bimodule. We say that $\mathcal{F}$ is *bicovariant* if we have a coaction $\gamma : \mathcal{F} \to \mathcal{F} \otimes \hat{M}$ compatible with the right coaction $\hat{\Delta}$ on $\hat{M}$ and satisfying

$$\gamma \pi_\ell = (\pi_\ell \otimes \iota)\hat{\Delta} \ .$$

The following result provides the bimodule version of Proposition 3.7.

Proposition 3.11. *Let $\mathcal{F}$ be a bicovariant W*-$\hat{M}$-$\hat{M}$-bimodule. Then, there exists a canonically defined Hilbert space K and a corepresentation $X \in M \otimes \mathrm{B}(K)$ such that $\mathcal{F} \simeq \hat{M} \otimes K$ as bicovariant W*-bimodules.*

Proof. Define $\mathcal{F}^\gamma = \{v \in \mathcal{F} \mid \gamma(v) = v \otimes 1\}$. Observe that $\langle v, w \rangle$ is invariant under $\hat{\Delta}$ and hence belongs to $\mathbb{C}$ whenever $v, w \in \mathcal{F}^\gamma$. So, $\mathcal{F}^\gamma$ is a Hilbert space. We shall show that $\mathcal{F}^\gamma$ is the Hilbert space that we are looking for and that the map $x \otimes \xi \mapsto \xi \cdot x$ extends to an isomorphism from $\hat{M} \otimes \mathcal{F}^\gamma$ onto $\mathcal{F}$.

Denote by $N = \mathcal{L}(\mathcal{F} \oplus \hat{M})$ the link algebra and denote by γ the right coaction of $(\hat{M}, \hat{\Delta})$ on N. Define

$$\theta : \hat{M} \to N : \theta(x) = \begin{pmatrix} \pi_\ell(x) & 0 \\ 0 & x \end{pmatrix} \ .$$

Then, $\gamma\theta = (\theta \otimes \iota)\hat{\Delta}$ by bicovariance of $\mathcal{E}$. From Proposition 1.22 in [27], it follows that γ is a dual coaction. This means that the formula

$$N^\gamma \to M \otimes N^\gamma : z \mapsto (\iota \otimes \theta)(W^*)(1 \otimes z)(\iota \otimes \theta)(W)$$

defines a left coaction of (M, Δ) on the fixed point algebra

$$N^\gamma = \{x \in N \mid \gamma(x) = x \otimes 1\}$$

such that N is isomorphic with the crossed product $\hat{M} \ltimes N^\gamma$ and γ is the dual coaction.

If we now observe that

$$N^\gamma = \begin{pmatrix} \mathcal{L}(\mathcal{F})^\gamma & \mathcal{F}^\gamma \\ (\mathcal{F}^\gamma)^* & \mathbb{C} \end{pmatrix}$$

we have found a corepresentation of (M, Δ) on the Hilbert space $\mathcal{F}^\gamma$ such that $\mathcal{F} \simeq \hat{M} \otimes \mathcal{F}^\gamma$ as bicovariant W*-bimodules. $\qquad \square$

4 Induction of corepresentations

We present a new approach to induction of unitary corepresentations of l.c. quantum groups, which works as well for the induction of corepresentations on C*-modules. We first provide some general machinery and start the induction procedure after Definition 4.4.

Let (M_1, Δ_1) be a closed quantum subgroup of (M, Δ) through the morphism $(M, \Delta) \xrightarrow{\pi} (M_1, \Delta_1)$. So, we have a normal, faithful *-homomorphism $\hat{\pi} : \hat{M}_1 \to \hat{M}$ satisfying $\hat{\Delta}\hat{\pi} = (\hat{\pi} \otimes \hat{\pi})\hat{\Delta}_1$.

Associated with π we have the coaction $\alpha : M \to M \otimes M_1$ which is formally given by $\alpha = (\iota \otimes \pi)\Delta$ and which, more precisely, satisfies

$$(\alpha \otimes \iota)(W) = W_{13}(\iota \otimes \hat{\pi})(W_1)_{23} .$$

Using the modular conjugations, we define as well $\hat{\pi}' : \hat{M}_1' \to \hat{M}'$ by

$$\hat{\pi}' : \hat{M}_1' \to \hat{M}' : \hat{\pi}'(x) = \hat{J}\pi(\hat{J}_1 x \hat{J}_1)\hat{J} \quad \text{for all} \quad x \in \hat{M}_1' .$$

Definition 4.1. We define $Q = M^\alpha := \{x \in M \mid \alpha(x) = x \otimes 1\}$. The von Neumann algebra Q should be considered as the *measured quantum homogeneous space*.

Observe that $\Delta(Q) \subset M \otimes Q$. Hence, the restriction of Δ to Q defines a left coaction of (M, Δ) on Q. By definition we have

$$\hat{M} \ltimes Q = \big(\Delta(Q) \cup \hat{M} \otimes 1 \big)'' .$$

Observing that $V^*(\hat{M} \ltimes Q)V = \big(Q \cup \hat{M} \big)'' \otimes 1$, we get that

$$\hat{M} \ltimes Q \cong \big(Q \cup \hat{M} \big)'' = (\hat{\pi}'(\hat{M}_1'))' .$$

We will often identify $\hat{M} \ltimes Q$ with its image in $\mathrm{B}(H)$.

Definition 4.2. Define

$$\mathcal{I} = \{v \in \mathrm{B}(H_1, H) \mid vx = \hat{\pi}'(x)v \ \text{for all} \ x \in \hat{M}_1'\} .$$

Defining $\langle v, w \rangle = v^*w$, the space $\mathcal{I}$ becomes a W*-$\hat{M}_1$-module. Since $\hat{M} \ltimes Q = (\hat{\pi}'(\hat{M}_1'))'$, we get that $\mathcal{I}$ is a W*-$(\hat{M} \ltimes Q)$-$\hat{M}_1$-*imprimitivity bimodule*.

Remark 4.3. So, we conclude that it is an almost trivial fact that the von Neumann algebras $\hat{M}_1$ and $\hat{M} \ltimes Q$ are Morita equivalent in a von Neumann algebraic sense. An important part of the present paper is to define the *locally compact quantum homogeneous space* $D \subset Q$ such that $\hat{A}_1$ is Morita equivalent with $\hat{A} \ _r\ltimes D$ (and such that $\hat{A}_1^u$ is Morita equivalent with $\hat{A}^u \ _f\ltimes D$). In order to do so, we will make use all the time of the von Neumann algebraic Morita equivalence $\mathcal{I}$.

Definition 4.4. Define

$$\alpha_{\mathcal{I}} : \mathcal{I} \to \mathcal{I} \otimes \hat{M} : \alpha_{\mathcal{I}}(v) = \hat{V}(v \otimes 1)(\iota \otimes \hat{\pi})(\hat{V}_1^*) .$$

Then, $\alpha_{\mathcal{I}}$ is a coaction of $(\hat{M}, \hat{\Delta})$ on $\mathcal{I}$ which is compatible with the coaction $\hat{\Delta}_1$ on $\hat{M}_1$. Moreover, if we equip $\hat{M} \ltimes Q$ with the dual coaction of $(\hat{M}, \hat{\Delta})$, the right module action of $\hat{M} \ltimes Q$ on $\mathcal{I}$ is covariant.

We now start the induction procedure. Let a corepresentation X of (M_1, Δ_1) on a C*-B-module $\mathcal{E}$ be given. So, $X \in \mathcal{L}(A_1 \otimes \mathcal{E})$ and $(\Delta_1 \otimes \iota)(X) = X_{13}X_{23}$.

Consider the C*-B-module $H \otimes \mathcal{E}$. We want to define a strict *-homomorphism $\pi_\ell : \hat{M}_1 \to \mathcal{L}(H \otimes \mathcal{E})$ which is formally given by the formula $\pi_\ell = (\hat{\pi} \otimes \theta)\hat{\Delta}_1^{\mathrm{op}}$, where $\theta : \hat{A}_1^{\mathrm{u}} \to \mathcal{L}(\mathcal{E})$ is the *-homomorphism associated with the corepresentation X.

Lemma 4.5. *There exists a unique strict *-homomorphism $\pi_\ell : \hat{M}_1 \to \mathcal{L}(H \otimes \mathcal{E})$ satisfying*

$$(\iota \otimes \pi_\ell)(W_1) = (\iota \otimes \hat{\pi})(W_1)_{12}\, X_{13} \ .$$

Proof. We would like to define $\pi_\ell(a) = (\hat{\pi} \otimes \iota)(X(a \otimes 1)X^*)$. This is somehow delicate, since we would have to give a meaning to $X(a \otimes 1)X^*$ belonging to $\hat{M}_1 \otimes \mathcal{L}(\mathcal{E})$ and to define $(\hat{\pi} \otimes \iota)$. We circumvent by defining

$$\pi_\ell : \hat{M}_1 \to \mathcal{L}(H \otimes \mathcal{E}) : \pi_\ell(a)(v \otimes 1)\xi = (v \otimes 1)X(a \otimes 1)X^*\xi$$

for every $\xi \in H_1 \otimes \mathcal{E}$ and every $v \in \mathrm{B}(H_1, H)$ satisfying $vx = \hat{\pi}(x)v$ for all $x \in \hat{M}_1$. It is not hard to check that $\pi_\ell(a)$ is a well defined operator in $\mathcal{L}(H \otimes \mathcal{E})$ and that π_ℓ is a strict *-homomorphism. $\qquad\square$

Equipped with $\pi_\ell : \hat{M}_1 \to \mathcal{L}(H \otimes \mathcal{E})$ together with $\pi_r : \hat{M} \to \mathcal{L}(H \otimes \mathcal{E}) : x \mapsto \hat{J}x^*\hat{J} \otimes 1$ and $M' \to \mathcal{L}(H \otimes \mathcal{E}) : y \mapsto y \otimes 1$, we have translated the unitary corepresentation $X \in \mathcal{L}(A_1 \otimes \mathcal{E})$ into a *bicovariant B-correspondence*

$$\underset{\hat{M}_1}{}\boxed{\begin{matrix} M' \\ H \otimes \mathcal{E} \end{matrix}}_{\hat{M}} \ . \tag{4.1}$$

The bicovariance of the above B-correspondence can also be expressed by the coaction

$$\alpha_{H \otimes \mathcal{E}} : H \otimes \mathcal{E} \to \mathcal{M}(H \otimes \mathcal{E} \otimes \hat{A}) : \xi \mapsto \hat{V}_{13}(\xi \otimes 1)$$

which is compatible with the trivial coaction on B.

We can now use Definition 12.9 and Proposition 12.13 to define

- the C*-B-module $\widetilde{\mathcal{F}} = \mathcal{I} \underset{\pi_\ell}{\otimes} (H \otimes \mathcal{E})$;

- a left module action and a right module action such that we get a B-correspondence $\underset{\hat{M} \ltimes Q}{}\boxed{\widetilde{\mathcal{F}}}_{\hat{M}}$;

- the product coaction $\alpha_{\widetilde{\mathcal{F}}}$ of $\alpha_{\mathcal{I}}$ and $\alpha_{H \otimes \mathcal{E}}$.

The product coaction $\alpha_{\widetilde{\mathcal{F}}}$ is compatible with the trivial coaction on B and hence yields a corepresentation $Y \in \mathcal{L}(\widetilde{\mathcal{F}} \otimes \hat{A})$. By construction (see Remark 3.6), we then get the bicovariant B-correspondence

$$\underset{\hat{M}}{}\boxed{\begin{matrix} M' \\ \widetilde{\mathcal{F}} \end{matrix}}_{\hat{M}} \ .$$

By Proposition 3.7, we get a canonically determined C*-B-module $\mathrm{Ind}\,\mathcal{E}$ together with a unitary corepresentation $\mathrm{Ind}\,X \in \mathcal{L}(A \otimes \mathcal{E})$ such that

$$\underset{\hat{M}}{}\boxed{\begin{matrix} M' \\ \widetilde{\mathcal{F}} \end{matrix}}_{\hat{M}} \ \cong \ \underset{\hat{M}}{}\boxed{\begin{matrix} M' \\ H \otimes \mathrm{Ind}\,\mathcal{E} \end{matrix}}_{\hat{M}}$$

as bicovariant correspondences.

Since Q coincides with the fixed point algebra of $\hat{M} \ltimes Q$ under the dual coaction, we also get a strict *-homomorphism $\rho : Q \to \mathcal{L}(\mathrm{Ind}\,\mathcal{E})$ such that

$$\underset{\hat{M} \ltimes Q}{}\boxed{\begin{matrix} M' \\ \widetilde{\mathcal{F}} \end{matrix}}_{\hat{M}} \ \cong \ \underset{\hat{M} \ltimes Q}{}\boxed{\begin{matrix} M' \\ H \otimes \mathrm{Ind}\,\mathcal{E} \end{matrix}}_{\hat{M}}$$

where, on the right hand side, the left module action $\pi_\ell : \hat{M} \ltimes Q \to \mathcal{L}(H \otimes \operatorname{Ind}\mathcal{E})$ is determined by

$$(\iota \otimes \pi_\ell)(W) = W_{12}(\operatorname{Ind} X)_{13} \quad \text{and} \quad \pi_\ell(x) = 1 \otimes \rho(x) \ \text{ for all } x \in Q \ .$$

By construction we get the following covariance relation:

$$(\iota \otimes \rho)\Delta(x) = (\operatorname{Ind} X)^*(1 \otimes \rho(x))(\operatorname{Ind} X) \quad \text{for all } \ x \in \mathbb{Q} \ .$$

Hence, we obtain the expected result that the induced corepresentation comes with a covariant representation of the measured quantum homogeneous space.

Definition 4.6. The C*-B-module $\operatorname{Ind}\mathcal{E}$ is called the *induced C*-B-module* of $\mathcal{E}$ and the unitary corepresentation $\operatorname{Ind} X$ is called the *induced corepresentation* of X.

Let B and B_1 be C*-algebras. Let $\mathcal{E}$ be a C*-B-module and let $\mathcal{G}$ be a C*-B_1-module. Suppose that $\mu : B \to \mathcal{L}(\mathcal{G})$ is a non-degenerate *-homomorphism. Then, we have the interior tensor product $\mathcal{E} \underset{\mu}{\otimes} \mathcal{G}$ as a C*-B_1-module. Suppose now that $X \in \mathcal{L}(A_1 \otimes \mathcal{E})$ is a corepresentation of (A_1, Δ_1) on $\mathcal{E}$. Then we have $X \underset{\mu}{\otimes} 1$ as a corepresentation on $\mathcal{E} \underset{\mu}{\otimes} \mathcal{G}$. Since our construction of the induced corepresentation is completely natural, the following result is obvious.

Proposition 4.7. *We have* $\operatorname{Ind}(\mathcal{E} \underset{\mu}{\otimes} \mathcal{G}) \cong \operatorname{Ind}(\mathcal{E}) \underset{\mu}{\otimes} \mathcal{G}$ *and* $\operatorname{Ind}(X \underset{\mu}{\otimes} 1) = \operatorname{Ind}(X) \underset{\mu}{\otimes} 1$ *in a natural way.*

The representation of Q *on* $\operatorname{Ind}(\mathcal{E} \underset{\mu}{\otimes} \mathcal{G})$ *is intertwined with the representation* $Q \underset{\mu}{\otimes} 1$ *on* $\operatorname{Ind}(\mathcal{E}) \underset{\mu}{\otimes} \mathcal{G}$.

5 First imprimitivity theorem

In the previous section we defined the induced corepresentation $\operatorname{Ind} X$ of a corepresentation X of a closed quantum subgroup (M_1, Δ_1) of (M, Δ). Such an induced corepresentation comes with a covariant representation of the measured quantum homogeneous space $Q \to \mathcal{L}(\operatorname{Ind}\mathcal{E})$.

A natural question is now of course if an imprimitivity result holds. More precisely, let $Z \in \mathcal{L}(A \otimes \mathcal{F})$ be a corepresentation of (A, Δ) on a C*-B-module $\mathcal{F}$ and $\rho : Q \to \mathcal{L}(\mathcal{F})$ a strict *-homomorphism which is covariant. Does there exist a corepresentation X of (A_1, Δ_1) on a C*-B-module $\mathcal{E}$ such that

$$(\mathcal{F} \ , \ Z \ , \ \text{rep. of } Q) \ \cong \ (\operatorname{Ind}\mathcal{E} \ , \ \operatorname{Ind} X \ , \ \text{rep. of } Q) \ ?$$

It is quite clear that the answer is negative in general. If $M_1 = \mathbb{C}$, the one-point subgroup, it is obvious to check that $Q = M$ and that the induced corepresentations are the multiples of the regular corepresentation W of (M, Δ). The question in the previous paragraph becomes the following: is every pair (X, ρ) of a corepresentation X of (M, Δ) and a covariant representation ρ of M isomorphic with a multiple of the regular corepresentation of (M, Δ) and the standard representation of M on H? But, this question is equivalent with the question

$$\hat{A}^{\mathrm{u}} \ {}_f\!\ltimes A \cong \mathcal{K}(H) \ ?$$

This property is precisely the *strong regularity* of the quantum group (A, Δ), see Definition 2.11.

Conclusion: we can only hope for an imprimitivity result if strong regularity holds. Otherwise, imprimitivity already fails for the one-point subgroup.

Theorem 5.1 (First Imprimitivity Theorem). *Let (M, Δ) be a strongly regular locally compact quantum group. Let (M_1, Δ_1) be a closed quantum subgroup and $Q \subset M$ the measured quantum homogeneous space in the sense of Definition 4.1.*

A corepresentation $Z \in \mathcal{L}(A \otimes \mathcal{F})$ of (M, Δ) on the C*-B-module $\mathcal{F}$ is induced from a corepresentation of (M_1, Δ_1) if and only if there exists a strict *-homomorphism $\rho : Q \to \mathcal{L}(\mathcal{F})$ such that

$$(\iota \otimes \rho)\Delta(x) = Z^*(1 \otimes \rho(x))Z \quad \text{for all } \ x \in Q \ . \tag{5.1}$$

Proof. It is clear that we only have to prove one implication. So, let $Z \in \mathcal{L}(A \otimes \mathcal{F})$ be a corepresentation of (M, Δ) on the C*-B-module $\mathcal{F}$ and let $\rho : Q \to \mathcal{L}(\mathcal{F})$ be a strict *-homomorphism satisfying the covariance relation (5.1).

To obtain the C*-B-module $\mathcal{E}$ and a corepresentation of (M_1, Δ_1) on it, we perform exactly the inverse of the induction procedure, tensoring with the inverse of the W*-imprimitivity bimodule $\mathcal{I}$ defined in Definition 4.2.

We claim that there exists a unique normal *-homomorphism $\pi_\ell : \hat{M} \ltimes Q \to \mathcal{L}(H \otimes \mathcal{F})$ such that

$$(\iota \otimes \pi_\ell)(W) = W_{12} Z_{13} \quad \text{and} \quad \pi_\ell(x) = 1 \otimes \rho(x) \ \text{ for all } \ x \in Q .$$

Indeed, it suffices to define $\pi_\ell(z) = Z(\iota \otimes \rho)(V(z \otimes 1)V^*)Z^*$. For all $z \in \hat{M} \ltimes Q$, the element $V(z \otimes 1)V^*$ belongs to $B(H) \otimes Q$. From Lemma 12.12 we know that we can extend $\iota \otimes \rho$ to the von Neumann algebra $B(H) \otimes Q$. It is easy to check that π_ℓ, once well-defined, satisfies the required conditions.

Using the anti-homomorphism $\pi_r : \hat{M} \to \mathcal{L}(H \otimes \mathcal{F}) : \pi_r(x) = \hat{J}x^*\hat{J} \otimes 1$ and the homomorphism $M' \to \mathcal{L}(H \otimes \mathcal{F}) : y \mapsto y \otimes 1$, we get a bicovariant B- correspondence

$$_{\hat{M} \ltimes Q} \overset{M'}{\boxed{H \otimes \mathcal{F}}} {}_{\hat{M}} .$$

In Definition 4.2 we defined the W*-imprimitivity bimodule $\mathcal{I}$. We can define its inverse as

$$\mathcal{I}^* = \{v \in B(H, H_1) \mid xv = v\hat{\pi}'(x) \quad \text{for all } \ x \in \hat{M}_1'\} .$$

Then, $\mathcal{I}^*$ is a W*-$\hat{M}_1$-$\hat{M} \ltimes Q$-bimodule. We can again define, as in Definition 4.4 a coaction of $(\hat{M}, \hat{\Delta})$ on $\mathcal{I}^*$.

We can then define, using Definition 12.9 and Proposition 12.13,

- the C*-B-module $\widetilde{\mathcal{E}} = \mathcal{I}^* \underset{\pi_\ell}{\otimes} (H \otimes \mathcal{F})$,

- a left and a right module action such that we get a B-correspondence $_{\hat{M}_1} \boxed{\widetilde{\mathcal{E}}} {}_{\hat{M}}$,

- the product coaction $\alpha_{\widetilde{\mathcal{E}}}$ of $(\hat{M}, \hat{\Delta})$ on $\widetilde{\mathcal{E}}$.

Hence, we have a bicovariant B-correspondence

$$_{\hat{M}_1} \overset{M'}{\boxed{\widetilde{\mathcal{E}}}} {}_{\hat{M}} .$$

The homomorphism $M' \to \mathcal{L}(\widetilde{\mathcal{E}})$ and the anti-homomorphism $\hat{M} \to \mathcal{L}(\widetilde{\mathcal{E}})$ are covariant. From the strong regularity of (M, Δ) it follows that we find a canonically determined C*-B-module $\mathcal{E}$ such that $\widetilde{\mathcal{E}} \cong H \otimes \mathcal{E}$ where the isomorphism intertwines the homomorphism $M' \to \mathcal{L}(\widetilde{\mathcal{E}})$ with $x \mapsto x \otimes 1$ and the anti-homomorphism $\hat{M} \to \mathcal{L}(\widetilde{\mathcal{E}})$ with $y \mapsto \hat{J}y^*\hat{J} \otimes 1$.

Exactly as in the proof of Proposition 3.7 the homomorphism $\hat{M}_1 \to \mathcal{L}(\widetilde{\mathcal{E}})$ is intertwined with a homomorphism $\pi_\ell : \hat{M}_1 \to \mathcal{L}(H \otimes \mathcal{E})$ such that

$$(\iota \otimes \pi_\ell)(W_1) = (\iota \otimes \hat{\pi})(W_1)_{12} X_{13}$$

where $X \in \mathcal{L}(A_1 \otimes \mathcal{E})$ is a corepresentation of (M_1, Δ_1) on the B-module $\mathcal{E}$. We get

$$_{\hat{M}_1} \overset{M'}{\boxed{\widetilde{\mathcal{E}}}} {}_{\hat{M}} \quad \cong \quad _{\hat{M}_1} \overset{M'}{\boxed{H \otimes \mathcal{E}}} {}_{\hat{M}}$$

as bicovariant correspondences.

It remains to prove that $\mathcal{F} = \operatorname{Ind}\mathcal{E}$ and $Z = \operatorname{Ind}X$. For this it suffices to observe that the interior tensor product of $\mathcal{I}$ and $\mathcal{I}^*$ is canonically isomorphic with $\hat{M} \ltimes Q$ as a W*-$\hat{M} \ltimes Q$-bimodule equipped with the dual coaction. Hence,

$$H \otimes \operatorname{Ind}\mathcal{E} = \mathcal{I} \underset{\hat{M}_1}{\otimes} (H \otimes \mathcal{E}) = \mathcal{I} \underset{\hat{M}_1}{\otimes} \mathcal{I}^* \underset{\hat{M} \ltimes Q}{\otimes} (H \otimes \mathcal{F}) = (\hat{M} \ltimes Q) \underset{\hat{M} \ltimes Q}{\otimes} (H \otimes \mathcal{F}) = H \otimes \mathcal{F} \; .$$

$\square$

6 Quantum homogeneous spaces and Mackey imprimitivity

We fix a locally compact quantum group (M, Δ). *We suppose throughout this section that (M, Δ) is strongly regular.* We fix a closed quantum subgroup (M_1, Δ_1). Recall that $Q \subset M$ denotes the measured quantum homogeneous space.

We shall prove the following crucial results.

Theorem 6.1. *There exists a unique C*-subalgebra $D \subset Q$ satisfying*

- *D is strongly dense in Q,*

- *$\Delta(D) \subset \mathcal{M}(A \otimes D)$ and $\Delta : D \to \mathcal{M}(A \otimes D)$ is a continuous coaction of (A, Δ) on D,*

- *$\Delta(Q) \subset \mathcal{M}(\mathcal{K}(H) \otimes D)$ and the *-homomorphism $\Delta : Q \to \mathcal{L}(H \otimes D)$ is strict.*

We call D the quantum homogeneous space.

Theorem 6.2. *There exist canonical covariant Morita equivalences*

$$\hat{A} \; {}_r\!\ltimes D \underset{\text{Morita}}{\sim} \hat{A}_1 \quad and \quad \hat{A}^{\mathrm{u}} \; {}_f\!\ltimes D \underset{\text{Morita}}{\sim} \hat{A}_1^{\mathrm{u}} \; .$$

As we shall see, the uniqueness statement in Theorem 6.1 is not so hard to prove and valid without the assumption on strong regularity. Our existence proof of D uses the strong regularity assumption (in fact, regularity suffices) but it is not excluded that D even exists without regularity assumptions. We recall however that we cannot hope for an imprimitivity theorem in the non-regular case.

Remark 6.3. The precise meaning of the statement Theorem 6.2 is the following. There exist a C*-$\hat{A}_1^{\mathrm{u}}$-module $\mathcal{E}_f$, a natural isomorphism $\mathcal{K}(\mathcal{E}_f) \cong \hat{A}^{\mathrm{u}} \; {}_f\!\ltimes D$ and a right coaction of $(\hat{A}, \hat{\Delta})$ on $\mathcal{E}_f$ that is compatible with the coaction $(\iota \otimes \hat{\pi})\hat{\Delta}_1^{\mathrm{u}}$ on $\hat{A}_1^{\mathrm{u}}$ and coincides with the dual coaction on $\hat{A}^{\mathrm{u}} \; {}_f\!\ltimes D$. Composing the Morita equivalence $\mathcal{E}_f$ between $\hat{A}^{\mathrm{u}} \; {}_f\!\ltimes D$ and $\hat{A}_1^{\mathrm{u}}$ on one side with the regular representation, we get the Morita equivalence $\hat{A} \; {}_r\!\ltimes D \underset{\text{Morita}}{\sim} \hat{A}_1$.

The right coaction $\alpha : M \to M \otimes M_1$ of (M_1, Δ_1) on M by right translation (see the beginning of Section 4), restricts to a continuous right coaction of (A_1, Δ_1) on A. The reduced crossed product $A \rtimes_r \hat{A}^{\mathrm{op}}_1$ is given as the closed linear span of $\alpha(A)(1 \otimes \hat{J}_1 \hat{A}_1 \hat{J}_1)$.

Corollary 6.4. *There is a natural covariant Morita equivalence*

$$D \underset{\text{Morita}}{\sim} A \rtimes_r \hat{A}^{\mathrm{op}}_1 \; .$$

Proof. Observe that we have covariant isomorphisms

$$A \rtimes_r \hat{A}^{\mathrm{op}}_1 \cong [A \; \hat{J}\hat{\pi}(\hat{A}_1)\hat{J}] \cong [JAJ \; \hat{\pi}(\hat{A}_1)] \cong \hat{A}_1 \rtimes_r A^{\mathrm{op}} \; ,$$

where we consider the crossed product of $\hat{A}_1$ equipped with the right coaction $(\iota \otimes \hat{\pi})\hat{\Delta}_1$ of $(\hat{A}, \hat{\Delta})$.

Since we have a covariant Morita equivalence $\hat{A} \,_r\ltimes D \underset{\text{Morita}}{\sim} \hat{A}_1$, we can take the crossed product in order to obtain a covariant Morita equivalence

$$(\hat{A} \,_r\ltimes D) \rtimes_r A^{\mathrm{op}} \underset{\text{Morita}}{\sim} \hat{A}_1 \rtimes_r A^{\mathrm{op}} .$$

The biduality theorem gives a natural covariant Morita equivalence

$$(\hat{A} \,_r\ltimes D) \rtimes_r A^{\mathrm{op}} \underset{\text{Morita}}{\sim} D$$

and then we are done. $\qquad\square$

Remark 6.5. We know of course that whenever G_1 is a closed subgroup of a locally compact group G, the action of G_1 on G by right translations is proper. Hence, the full and reduced crossed products $C_0(G) \rtimes_f G_1$ and $C_0(G) \rtimes_r G_1$ coincide.

There are strong indications that the same result is no longer valid in general, even for strongly regular l.c. quantum groups. Nevertheless, if either (A, Δ) is co-amenable (which means that $A^{\mathrm{u}} = A$) or (A_1, Δ_1) is amenable, the full and reduced crossed products $A \rtimes_f \hat{A}^{\mathrm{u,op}}_1$ and $A \rtimes_r \hat{A}^{\mathrm{op}}_1$ coincide. The second part is of course obvious. So, suppose that (A, Δ) is co-amenable. Observe that this is for instance the case when $A = C_0(G)$ and hence, this case covers the group case of the previous paragraph. Using twice the co-amenability of (A, Δ), we have

$$A \rtimes_f \hat{A}^{\mathrm{u,op}}_1 \cong A^{\mathrm{u}} \rtimes_f \hat{A}^{\mathrm{u,op}}_1 \cong \hat{A}^{\mathrm{u}}_1 \rtimes_f A^{\mathrm{u,op}} \cong \hat{A}_1 \rtimes_f A^{\mathrm{u,op}} \cong \hat{A}_1 \rtimes_r A^{\mathrm{op}} \cong A \rtimes_r \hat{A}^{\mathrm{op}}_1 .$$

In general, although I do not have an explicit example, it might very well be that $(A, \alpha) \cong (A_1 \,_r\ltimes B, \hat{\beta})$, where $\beta : B \to \mathcal{M}(\hat{A}_1 \otimes B)$ is a sufficiently non-trivial reduced coaction and $\hat{\beta}$ is the dual right coaction of (A_1, Δ_1) on the crossed product. The statement $A \rtimes_f \hat{A}^{\mathrm{u,op}}_1 = A \rtimes_r \hat{A}^{\mathrm{op}}_1$ comes down to saying that β is as well a maximal coaction. There seems to be no reason why this should always be the case in a non-amenable, non-co-amenable situation.

In order to construct D we have to look at a covariant induction procedure. This has an independent interest. Indeed, we shall not only prove the Morita equivalence

$$\hat{A}^{\mathrm{u}} \,_f\ltimes D \underset{\text{Morita}}{\sim} \hat{A}^{\mathrm{u}}_1$$

but we also want that this Morita equivalence is covariant for a natural coaction of $(\hat{A}, \hat{\Delta})$, compatible with the dual coaction on $\hat{A}^{\mathrm{u}} \,_f\ltimes D$ and the comultiplication on $\hat{A}^{\mathrm{u}}_1$.

Let $\mathcal{E}$ be a C*-B-module. Suppose that we have a coaction $\beta : \mathcal{E} \to \mathcal{M}(\mathcal{E} \otimes \hat{A})$ compatible with a continuous coaction on B that we also denote by $\beta : B \to \mathcal{M}(B \otimes \hat{A})$.

Let $X \in \mathcal{L}(A_1 \otimes \mathcal{E})$ be a corepresentation of (M_1, Δ_1) on $\mathcal{E}$ satisfying the following compatibility relation with β:

$$(\iota \otimes \beta)\big(X(1 \otimes v)\big) = (\iota \otimes \hat{\pi})(W_1)_{13} X_{12}\big(1 \otimes \beta(v)\big) \quad \text{for all} \ v \in \mathcal{E} . \tag{6.1}$$

If we consider the *-homomorphism $\theta : \hat{A}^{\mathrm{u}}_1 \to \mathcal{L}(\mathcal{E})$ associated with the corepresentation X, Equation (6.1) becomes $\beta(\theta(a)v) = (\theta \otimes \hat{\pi})\hat{\Delta}_1(a)\beta(v)$ for all $a \in \hat{A}^{\mathrm{u}}_1$ and $v \in \mathcal{E}$.

Let $\mathcal{F} = \operatorname{Ind} \mathcal{E}$ be the induced C*-B-module with induced corepresentation $Y = \operatorname{Ind} X$ of (A, Δ) on $\mathcal{F}$. We shall construct an induced coaction $\operatorname{Ind} \beta$ on the induced C*-module $\operatorname{Ind} \mathcal{E}$.

Recall that, by definition of $\operatorname{Ind} \mathcal{E}$, we have an isomorphism

$$\Phi : \mathcal{I} \underset{\pi_\ell}{\otimes} (H \otimes \mathcal{E}) \to H \otimes \mathcal{F}$$

where $\pi_\ell : \hat{M}_1 \to \mathcal{L}(H \otimes \mathcal{E})$ is determined by $(\iota \otimes \pi_\ell)(W_1) = (\iota \otimes \hat{\pi})(W_1)_{12} X_{13}$.

Proposition 6.6. *There exists a unique coaction* $\mathrm{Ind}\,\beta : \mathrm{Ind}\,\mathcal{E} \to \mathcal{M}(\mathrm{Ind}\,\mathcal{E} \otimes \hat{A})$ *of* $(\hat{A}, \hat{\Delta})$ *on the induced* C^**-B-module* $\mathrm{Ind}\,\mathcal{E}$ *which is compatible with the coaction* β *on* B *and satisfies*

$$(\iota \otimes \mathrm{Ind}\,\beta)\Phi(v \underset{\pi_\ell}{\otimes} x) = W_{13}(\Phi \otimes \iota)\big((v \otimes 1) \underset{\pi_\ell \otimes \iota}{\otimes} W_{13}^*(\iota \otimes \beta)(x)\big) \quad \textit{for all } v \in \mathcal{I}, x \in H \otimes \mathcal{E}. \tag{6.2}$$

Writing γ *for* $\mathrm{Ind}\,\beta$*, writing* γ *as well for the associated coaction on* $\mathcal{K}(\mathrm{Ind}\,\mathcal{E})$ *and writing* $Y = \mathrm{Ind}\,X$*, we get that*

$$Q \subset \mathcal{L}(\mathrm{Ind}\,\mathcal{E})^\gamma \quad \textit{and} \quad (\iota \otimes \gamma)(Y) = W_{13}Y_{12}. \tag{6.3}$$

Remark that the defining Equation (6.2) makes sense as an equality in $\mathcal{M}(H \otimes \mathcal{E} \otimes H)$.

Proof. Write $\widetilde{\mathcal{E}} = H \otimes \mathcal{E}$. Define the coaction $\eta : \widetilde{\mathcal{E}} \to \mathcal{M}(\widetilde{\mathcal{E}} \otimes \hat{A})$ by $\eta(v) = W_{13}^*(\iota \otimes \beta)(v)$ and observe that η is compatible with the coaction β on B. We still write η for the associated coaction on $\mathcal{K}(\widetilde{\mathcal{E}})$. It is easily verified that $\eta(\pi_\ell(x)) = \pi_\ell(x) \otimes 1$ for all $x \in \hat{M}_1$.

If we equip $\mathcal{I}$ and $\hat{M}_1$ with the trivial coaction of $(\hat{M}, \hat{\Delta})$, the homomorphism $\pi_\ell : \hat{M}_1 \to \mathcal{L}(\widetilde{\mathcal{E}})$ is covariant in the sense of Definition 12.11. Hence, Proposition 12.13 yields a product coaction η_1 of $(\hat{A}, \hat{\Delta})$ on $\widetilde{\mathcal{F}} = \mathcal{I} \underset{\pi_\ell}{\otimes} \widetilde{\mathcal{E}}$ such that

$$\eta_1(v \underset{\pi_\ell}{\otimes} x) = (v \otimes 1) \underset{\pi_\ell \otimes \iota}{\otimes} W_{13}^*(\iota \otimes \beta)(x)$$

for all $v \in \mathcal{I}$ and $x \in \widetilde{\mathcal{E}}$.

We have an isomorphism $\Phi : \widetilde{\mathcal{F}} \to H \otimes \mathcal{F}$ and this allows to define $\eta_2 : H \otimes \mathcal{F} \to \mathcal{M}(H \otimes \mathcal{F} \otimes \hat{A})$ such that

$$\eta_2(\Phi(x)) = W_{13}(\Phi \otimes \iota)(\eta_1(x)) \quad \text{for all } x \in \widetilde{\mathcal{F}}.$$

We claim that η_2 is invariant under the right module action of $\hat{M}$ on $H \otimes \mathcal{F}$. Indeed, for $a \in \hat{M}$ and $x \in \widetilde{\mathcal{F}}$, we have

$$\eta_2\big((\hat{J}a^*\hat{J} \otimes 1)\Phi(x)\big) = \eta_2\big(\Phi(\pi_r(a)x)\big) = W_{13}(\Phi \otimes \iota)\big(\eta_1(\pi_r(a)x)\big).$$

We then observe that for $a \in \hat{M}$, $v \in \mathcal{I}$ and $y \in \widetilde{\mathcal{E}}$

$$\eta_1(\pi_r(a)(v \underset{\pi_\ell}{\otimes} y)) = (v \otimes 1) \underset{\pi_\ell \otimes \iota}{\otimes} W_{13}^*(\iota \otimes \beta)((\hat{J}a^*\hat{J} \otimes 1)y) = (v \otimes 1) \underset{\pi_\ell \otimes \iota}{\otimes} ((\hat{J} \otimes J)\hat{\Delta}^{\mathrm{op}}(a^*)(\hat{J} \otimes J))_{13}W_{13}^*(\iota \otimes \beta)(y).$$

We conclude that

$$W_{13}(\Phi \otimes \iota)\big(\eta_1(\pi_r(a)x)\big) = W_{13}((\hat{J} \otimes J)\hat{\Delta}^{\mathrm{op}}(a^*)(\hat{J} \otimes J))_{13}(\Phi \otimes \iota)\eta_1(x) = (\hat{J}a^*\hat{J} \otimes 1 \otimes 1)\eta_2(\Phi(x)).$$

This proves our claim.

On the other hand, in the construction of the induced module $\mathrm{Ind}\,\mathcal{E}$, we used the product coaction $\alpha_{\widetilde{\mathcal{F}}}$ of $\alpha_{\mathcal{I}}$ and $\alpha_{H \otimes \mathcal{E}}$. It is clear that the coactions $\alpha_{\widetilde{\mathcal{F}}}$ and η_1 commute. This implies that η_2 is invariant under the representation $M' \otimes 1$ of M' on $H \otimes \mathcal{F}$.

So, we have shown that η_2 is invariant under $M' \otimes 1$ as well as $\hat{M}' \otimes 1$. Hence, there exists a non-degenerate linear map $\gamma : \mathcal{F} \to \mathcal{M}(\mathcal{F} \otimes \hat{A})$ such that $\eta_2 = \iota \otimes \gamma$. Since η_1 is a coaction, the map $x \mapsto W_{13}^*(\iota \otimes \gamma)(x)$ defines a coaction of $(\hat{A}, \hat{\Delta})$ on $H \otimes \mathcal{F}$. This implies that γ is as well a coaction of $(\hat{A}, \hat{\Delta})$ on $\mathcal{F}$. We define $\mathrm{Ind}\,\beta := \gamma$.

By definition η_1 is invariant under the left module action of $\hat{M} \ltimes Q$ on $\widetilde{\mathcal{F}}$. It is then clear that γ satisfies (6.3). $\qquad\square$

We are now ready to prove the main Theorems 6.1 and 6.2

Proof of Theorem 6.1. Consider the C*-$\hat{A}_1$-module $\hat{A}_1$ equipped with the regular corepresentation $W_1 \in \mathcal{M}(A_1 \otimes \hat{A}_1)$ and the coaction $\beta := (\iota \otimes \hat{\pi})\hat{\Delta}_1$ of $(\hat{A}, \hat{\Delta})$.

Define the C*-$\hat{A}_1$-module $\mathcal{J} = \operatorname{Ind} \hat{A}_1$ together with the induced corepresentation $X \in \mathcal{M}(A \otimes \mathcal{J})$, the strict *-homomorphism $\theta : Q \to \mathcal{L}(\mathcal{J})$ and the induced coaction $\gamma = \operatorname{Ind} \beta$ of $(\hat{A}, \hat{\Delta})$ on $\mathcal{J}$ as in Proposition 6.6.

Continue writing γ for the coaction of $(\hat{A}, \hat{\Delta})$ on $\mathcal{K}(\mathcal{J})$. Then, we have a strict *-homomorphism $\theta : Q \to \mathcal{L}(\mathcal{J})$ such that $\gamma(\theta(x)) = \theta(x) \otimes 1$ for all $x \in Q$. We also have $(\iota \otimes \gamma)(X) = W_{13}X_{12}$.

We claim that $\theta : Q \to \mathcal{L}(\mathcal{J})^\gamma$ is a *-isomorphism. Using the regular representation $\hat{A}_1 \to \mathrm{B}(H_1)$, we get, using Proposition 4.7 that
$$\mathcal{J} \underset{\hat{A}_1}{\otimes} H_1 = \operatorname{Ind}(\hat{A}_1) \underset{\hat{A}_1}{\otimes} H_1 = \operatorname{Ind}(H_1) = H .$$

It is straightforward to check that, under these identifications,
$$\theta(x) \underset{\hat{A}_1}{\otimes} 1 = x \quad \text{for all } x \in Q$$

and $X \underset{\hat{A}_1}{\otimes} 1 = W$. So, we get an injective *-homomorphism $\mathcal{K}(\mathcal{J}) \to \hat{M} \ltimes Q$ which intertwines the coaction γ with the dual coaction on $\hat{M} \ltimes Q$. Since the fixed point algebra of $\hat{M} \ltimes Q$ under the dual coaction is precisely Q, we have proved our claim.

Combining Theorem 6.7 and Remark 6.8 with the isomorphism $\mathcal{L}(\mathcal{J})^\gamma \cong Q$, we conclude that there exists a strongly dense C*-subalgebra $D \subset Q = \mathcal{L}(\mathcal{J})^\gamma$ such that

- $\Delta : D \to \mathcal{M}(A \otimes D)$ is a continuous coaction of (A, Δ) on D;

- $\Delta : Q \to \mathcal{M}(\mathcal{K}(H) \otimes D)$ is well defined and strict.

So, we have proved the existence part of Theorem 6.1.

To prove uniqueness, suppose that D_1 and D_2 satisfy the conditions in the theorem. Then, we get that $D_1 = [(\omega \otimes \iota)\Delta(D_1) \mid \omega \in \mathrm{B}(H)_*]$ by continuity of the coaction. Using the strictness of $\Delta : Q \to \mathcal{M}(\mathcal{K}(H) \otimes D_1)$ and the fact that D_1 as well as D_2 are dense in Q, we obtain that

$$D_1 = [D_1 D_1] = [(\omega \otimes \iota)\big(\Delta(D_1)(\mathcal{K}(H) \otimes D_1)\big) \mid \omega \in \mathrm{B}(H)_*] = [(\omega \otimes \iota)\big(\Delta(Q)(\mathcal{K}(H) \otimes D_1)\big) \mid \omega \in \mathrm{B}(H)_*]$$
$$= [(\omega \otimes \iota)\big(\Delta(D_2)(\mathcal{K}(H) \otimes D_1)\big) \mid \omega \in \mathrm{B}(H)_*] = [D_2 D_1] .$$

By symmetry, we find that $D_2 = [D_1 D_2]$. Taking the adjoint, this gives $D_2 = [D_2 D_1]$ and we conclude that $D_1 = D_2$. $\qquad\square$

Proof of Theorem 6.2. The statement $\hat{A} \,_r\!\ltimes D \underset{\text{Morita}}{\sim} \hat{A}_1$ has already been shown in the proof of Theorem 6.1.

Loosely speaking, the first imprimitivity theorem 5.1 tells us that representations of $\hat{A}_1^{\mathrm{u}}$ on C*-B-modules are in one to one correspondence to covariant representations of the pair $(\hat{A}^{\mathrm{u}}, Q)$ on C*-B-modules. In order to prove the theorem, it suffices to show that

$$\text{any rep. of } \hat{A}^{\mathrm{u}} \,_f\!\ltimes D \text{ on a C*-}B\text{-module } \mathcal{F} \text{ extends to a strict *-homomorphism } Q \to \mathcal{L}(\mathcal{F}). \qquad (6.4)$$

We shall show this statement at the end of the proof.

Consider the C*-$\hat{A}_1^{\mathrm{u}}$-module $\hat{A}_1^{\mathrm{u}}$ equipped with the universal corepresentation $W_1^{\mathrm{u}} \in \mathcal{M}(A_1 \otimes \hat{A}_1^{\mathrm{u}})$ and the coaction $\beta := (\iota \otimes \hat{\pi})\hat{\Delta}_1^{\mathrm{u}}$ of $(\hat{A}, \hat{\Delta})$.

Define the C*-$\hat{A}_1^{\mathrm{u}}$-module $\mathcal{E}_f = \operatorname{Ind} \hat{A}_1^{\mathrm{u}}$ together with the induced corepresentation $X \in \mathcal{M}(A \otimes \mathcal{E}_f)$, the strict *-homomorphism $\theta : Q \to \mathcal{L}(\mathcal{E}_f)$ and the induced coaction $\gamma = \operatorname{Ind} \beta$ of $(\hat{A}, \hat{\Delta})$ on $\mathcal{E}_f$ as in Proposition 6.6. By the covariance of X and θ, we have an associated representation $\hat{A}^{\mathrm{u}} \,_f\!\ltimes D \to \mathcal{L}(\mathcal{E}_f)$, which intertwines the dual coaction on $\hat{A}^{\mathrm{u}} \,_f\!\ltimes D$ with the coaction γ on $\mathcal{E}_f$. We claim that this isomorphism identifies $\hat{A}^{\mathrm{u}} \,_f\!\ltimes D$ with $\mathcal{K}(\mathcal{E}_f)$.

Consider the C*-($\hat{A}^{\mathrm{u}}$ $_f\ltimes D$)-module $\hat{A}^{\mathrm{u}}$ $_f\ltimes D$. Using (6.4), we get a covariant representation of $\hat{A}^{\mathrm{u}}$ and Q in $\mathcal{M}(\hat{A}^{\mathrm{u}}$ $_f\ltimes D) = \mathcal{L}(\hat{A}^{\mathrm{u}}$ $_f\ltimes D)$. By the first imprimitivity theorem 5.1, we get a C*-($\hat{A}^{\mathrm{u}}$ $_f\ltimes D$)-module $\mathcal{E}'_f$ together with a representation of $\hat{A}^{\mathrm{u}}_1$ on $\mathcal{E}'_f$ such that $\hat{A}^{\mathrm{u}}$ $_f\ltimes D = \mathrm{Ind}\,\mathcal{E}'_f$. Then, by Proposition 4.7, we get

$$\mathcal{E}_f \underset{\hat{A}^{\mathrm{u}}_1}{\otimes} \mathcal{E}'_f = (\mathrm{Ind}\,\hat{A}^{\mathrm{u}}_1) \underset{\hat{A}^{\mathrm{u}}_1}{\otimes} \mathcal{E}'_f = \mathrm{Ind}(\hat{A}^{\mathrm{u}}_1 \underset{\hat{A}^{\mathrm{u}}_1}{\otimes} \mathcal{E}'_f) = \mathrm{Ind}\,\mathcal{E}'_f = \hat{A}^{\mathrm{u}} {}_f\ltimes D .$$

Conversely,

$$\mathrm{Ind}(\mathcal{E}'_f \underset{\hat{A}\,_f\ltimes D}{\otimes} \mathcal{E}_f) = (\mathrm{Ind}\,\mathcal{E}'_f) \underset{\hat{A}\,_f\ltimes D}{\otimes} \mathcal{E}_f = (\hat{A}^{\mathrm{u}} {}_f\ltimes D) \underset{\hat{A}\,_f\ltimes D}{\otimes} \mathcal{E}_f = \mathcal{E}_f = \mathrm{Ind}(\hat{A}^{\mathrm{u}}_1) .$$

From this it follows that $\mathcal{E}'_f \underset{\hat{A}\,_f\ltimes D}{\otimes} \mathcal{E}_f = \hat{A}^{\mathrm{u}}_1$. Since we have found the inverse module $\mathcal{E}'_f$, our claim is proved.

It remains to prove (6.4). So, let $Z \in \mathcal{L}(A \otimes \mathcal{F})$ be a corepresentation of (A, Δ) on a C*-B-module $\mathcal{F}$ and let $\rho : D \to \mathcal{L}(F)$ be a non-degenerate *-homomorphism which is covariant in the sense that

$$(\iota \otimes \rho)\Delta(x) = Z^*(1 \otimes \rho(x))Z \quad \text{for all } x \in D .$$

Since $\Delta : Q \to \mathcal{M}(\mathcal{K}(H) \otimes D)$ is strict, we can define a strict *-homomorphism

$$\mu : Q \to \mathcal{L}(H \otimes \mathcal{F}) : \mu(x) = Z(\iota \otimes \rho)\Delta(x)Z^* .$$

But then, $\mu(x) = 1 \otimes \rho(x)$ for all $x \in D$. Since D is dense in Q and since μ is strict, it follows that there exists a strict *-homomorphism $\eta : Q \to \mathcal{L}(\mathcal{F})$ such that $\mu(z) = 1 \otimes \eta(z)$ for all $z \in Q$. Then, η is the extension of ρ that we were looking for. $\square$

The major tool used in the proof of Theorem 6.1 is the characterization of reduced crossed products. This is a quantum version of a theorem of Landstad (Theorem 3 in [19]).

Theorem 6.7. *Let (A, Δ) be a locally compact quantum group. Let $\beta : B \to \mathcal{M}(B \otimes \hat{A})$ be a reduced continuous coaction of $(\hat{A}, \hat{\Delta})$ on the C*-algebra B. Then, the following conditions are equivalent.*

   a) *There exists a C*-algebra D and a continuous coaction $\alpha : D \to \mathcal{M}(A \otimes D)$ such that $(B, \beta) \cong (\hat{A} {}_r\ltimes D, \hat{\alpha})$, where $\hat{\alpha}$ denotes the dual coaction.*

   b) *There exists a corepresentation $X \in \mathcal{M}(A \otimes B)$ of (A, Δ) in B which is covariant in the sense that*

$$(\iota \otimes \beta)(X) = W_{13}X_{12} .$$

If the second condition is fulfilled, D can be taken as the unique C-subalgebra of $\mathcal{M}(B)^\beta$ satisfying*

   • *the map $\alpha : x \mapsto X^*(1 \otimes x)X$ defines a continuous coaction of (A, Δ) on D;*

   • *B is the closed linear span of $\{x(\omega \otimes \iota)(X) \mid x \in D, \omega \in \mathrm{B}(H)_*\}$.*

*Moreover, an explicit *-isomorphism $B \to \hat{A} {}_r\ltimes D$ is then given by $\eta : z \mapsto X^*\beta(z)_{21}X$.*

Proof. In the course of this proof, we denote by $[X]$ the closed linear span of a subset of a C*-algebra. Suppose first that $\alpha : D \to \mathcal{M}(A \otimes D)$ is a continuous coaction of (A, Δ) on the C*-algebra D. Then, the crossed product $\hat{A} {}_r\ltimes D$ is defined as $[\alpha(D)(\hat{A} \otimes 1)]$. It is clear that we can take $X = W \otimes 1 \in \mathcal{M}(A \otimes (\hat{A} {}_r\ltimes D))$.

Suppose next that the second condition is fulfilled. If we write $\eta : B \to \mathcal{M}(\mathcal{K} \otimes B) : \eta(z) = X^*\beta(z)_{21}X$, we observe that $(\iota \otimes \eta)(X) = W \otimes 1$. Hence, there is a uniquely defined faithful non-degenerate *-homomorphism $\theta : \hat{A} \to \mathcal{M}(B)$ such that $X = (\iota \otimes \theta)(W)$. Moreover $\beta\theta = (\theta \otimes \iota)\hat{\Delta}$.

We first prove the uniqueness statement. Suppose that $D \subset \mathcal{M}(B)^\beta$ is a C*-algebra that satisfies both conditions in the theorem. Then,

$$[(\omega \otimes \iota)\eta(z) \mid z \in B, \omega \in \mathrm{B}(H)_*] = [(\omega \otimes \iota)\eta(x\theta(a)) \mid x \in D, a \in \hat{A}, \omega \in \mathrm{B}(H)_*]$$
$$= [(\omega \otimes \iota)\big(X^*(1 \otimes x)X(a \otimes 1)\big) \mid x \in D, a \in \hat{A}, \omega \in \mathrm{B}(H)_*]$$
$$= [(\omega \otimes \iota)\big(X^*(1 \otimes x)X\big) \mid x \in D, \omega \in \mathrm{B}(H)_*] = D .$$

Since the left hand side does not depend on D, uniqueness of D has been proved.

In order to prove existence of D, define

$$D := [(\omega \otimes \iota)\eta(z) \mid z \in B, \omega \in \mathrm{B}(H)_*] .$$

We first show that D is a C*-algebra. Since the coaction β of $(\hat{A}, \hat{\Delta})$ on B is continuous, we get that $[\beta(B)(1 \otimes JAJ)] = [(1 \otimes JAJ)\beta(B)]$, since this space is exactly the crossed product of B with the coaction β. By regularity of (A, Δ), we also know that $\mathcal{K}(H) = [JAJ \, \hat{A}]$. So,

$$[\eta(B)(\mathcal{K}(H) \otimes 1)\eta(B)] = [\eta(B)(JAJ \, \hat{A} \otimes 1)\eta(B)] = [X^*\big(\beta(B)(1 \otimes JAJ)\big)_{21} X(\hat{A} \otimes 1)X^* \, \beta(B)_{21}X]$$
$$= [(JAJ \otimes 1)X^*\big(\beta(B) \, \beta(\theta(\hat{A})) \, \beta(B)\big)_{21}X] = [(JAJ \otimes 1)\eta(B)] .$$

Applying $(\omega \otimes \iota)$ on both sides of this equality, we obtain that $D = [DD]$. Hence, D is a C*-subalgebra of $\mathcal{M}(B)^\beta$.

Define $\alpha : D \to \mathcal{M}(A \otimes B) : \alpha(x) = X^*(1 \otimes x)X$. Then,

$$[\alpha(D)(A \otimes 1)] = [(\omega \otimes \iota \otimes \iota)(X_{23}^* X_{13}^* \beta(B)_{31} X_{13} X_{23})(A \otimes 1) \mid \omega \in \mathrm{B}(H)_*]$$
$$= [(\omega \otimes \iota \otimes \iota)\big(V_{12}X_{13}^* V_{12}^* \, \beta(B)_{31} \, V_{12}X_{13}V_{12}^*\big)(A \otimes 1) \mid \omega \in \mathrm{B}(H)_*]$$
$$= [(\omega \otimes \iota \otimes \iota)\big(V_{12}\eta(B)_{13}V_{12}^*(\mathcal{K}(H) \otimes A \otimes 1)\big) \mid \omega \in \mathrm{B}(H)_*]$$
$$= [(\omega \otimes \iota \otimes \iota)\big(V_{12}\eta(B)_{13}(1 \otimes A \otimes 1) \mid \omega \in \mathrm{B}(H)_*]$$

because $V \in \mathcal{M}(\mathcal{K}(H) \otimes A)$. From the regularity of V, it follows that $[(\mathcal{K}(H) \otimes 1)V(1 \otimes A)] = \mathcal{K}(H) \otimes A$ and hence,

$$[\alpha(D)(A \otimes 1)] = A \otimes [(\omega \otimes \iota)\eta(B) \mid \omega \in \mathrm{B}(H)_*] = A \otimes D .$$

So, α defines a continuous coaction of (A, Δ) on D.

Further, since X is a corepresentation, we know that $X \in \mathcal{M}(A \otimes \theta(\hat{A}))$. So, by the continuity of the coaction β, we get,

$$[D\theta(\hat{A})] = [(\omega \otimes \iota)\big(X^*\beta(z)_{21}X(\mathcal{K}(H) \otimes \theta(\hat{A}))\big) \mid \omega \in \mathrm{B}(H)_*]$$
$$= [(\omega \otimes \iota)\big(X^*\beta(z)_{21}(\mathcal{K}(H) \otimes \theta(\hat{A}))\big) \mid \omega \in \mathrm{B}(H)_*]$$
$$= [(\omega \otimes \iota)\big(X^*(\mathcal{K}(H) \otimes B\theta(\hat{A}))\big) \mid \omega \in \mathrm{B}(H)_*] = B .$$

This ends the proof of the theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 6.8. There is another way to characterize uniquely the C*-algebra D. We claim that D is the unique C*-subalgebra of $\mathcal{M}(B)^\beta$ that satisfies the following conditions.

- The map $\alpha : x \mapsto X^*(1 \otimes x)X$ defines a continuous coaction of (A, Δ) on D.

- The map $\alpha : \mathcal{M}(B)^\beta \to \mathcal{M}(\mathcal{K}(H) \otimes D) : z \mapsto X^*(1 \otimes z)X$ is well defined and continuous on the unit ball of $\mathcal{M}(B)^\beta$ if we equip $\mathcal{M}(B)^\beta$ with the strict topology inherited from $\mathcal{M}(B)$ and $\mathcal{M}(\mathcal{K}(H) \otimes D)$ with the strict topology.

- $D \subset \mathcal{M}(B)^\beta$ is non-degenerate in the sense that $B = [DB]$.

First observe that the C*-algebra D defined above satisfies these conditions. Since D obviously satisfies the first and third condition, it remains to prove the second condition. But $\eta : B \to \hat{A}\ _r\!\ltimes D$ is a *-isomorphism and since the inclusion $\hat{A}\ _r\!\ltimes D \to \mathcal{M}(\mathcal{K}(H) \otimes D)$ is non-degenerate, we get a strictly continuous map $\eta : \mathcal{M}(B) \to \mathcal{M}(\mathcal{K}(H) \otimes D)$. It suffices to restrict η to $\mathcal{M}(B)^\beta$.

We prove the uniqueness: suppose that D_1 and D_2 satisfy the stated conditions. Then,

$$[D_1 D_2] = [(\omega \otimes \iota)(\alpha(D_1))\, D_2 \mid \omega \in \mathrm{B}(H)_*] \subset [(\omega \otimes \iota)(\alpha(\mathcal{M}(B)^\beta))\, D_2 \mid \omega \in \mathrm{B}(H)_*] \subset D_2 \ .$$

On the other hand, let (e_i) be a bounded approximate identity for the C*-algebra D_1. Since $D_1 \subset \mathcal{M}(B)^\beta$ is non-degenerate, we get that (e_i) is a net in $\mathcal{M}(B)^\beta$ that converges to 1 in the strict topology of $\mathcal{M}(B)$. Take $\omega \in \mathrm{B}(H)_*$ such that $\omega(1) = 1$. Then, $\big((\omega \otimes \iota)\alpha(e_i)\big)_i$ is a net in $\mathcal{M}(D_2)$ that converges strictly to 1. It follows that $D_2 \subset [D_1 D_2]$ because $(\omega \otimes \iota)\alpha(e_i) \in D_1$ for all i. We conclude that $D_2 = [D_1 D_2]$. By symmetry, we get $[D_2 D_1] = D_1$. Taking the adjoint, we find that $D_1 = D_2$. This proves our claim.

When proving Theorem 6.1 we have in a natural way that $D \subset Q$ covariantly, where Q is a von Neumann algebra on which (M, Δ) coacts. Moreover, we have that $\hat{A}\ _r\!\ltimes D$ is a dense subalgebra of $\hat{M} \ltimes Q$. We claim that this implies that D is dense in Q. Indeed, if we denote by $\hat{\alpha} : \hat{A}\ _r\!\ltimes D \to \mathcal{M}((\hat{A}\ _r\!\ltimes D) \otimes \hat{A})$ the dual coaction, we have

$$\alpha(D) = [(\iota \otimes \iota \otimes \omega)\big(\hat{W}_{13}\hat{\alpha}(\hat{A}\ _r\!\ltimes D)\hat{W}_{13}^*\big) \mid \omega \in \mathrm{B}(H)_*] \ .$$

Hence, the σ-weak closure of $\alpha(D)$ is equal to the σ-weak closure of

$$[(\iota \otimes \iota \otimes \omega)\big(\hat{W}_{13}\hat{\alpha}(\hat{M} \ltimes Q)\hat{W}_{13}^*\big) \mid \omega \in \mathrm{B}(H)_*]$$

and so, equal to the σ-weak closure of $[\alpha((\omega \otimes \iota)\alpha(Q)) \mid \omega \in \mathrm{B}(H)_*]$. Since $\alpha(Q)(\mathrm{B}(H) \otimes 1)$ is σ-weakly dense in $\mathrm{B}(H) \otimes Q$, we conclude that D is σ-weakly dense in Q.

7 Induction of coactions and Green imprimitivity

We fix a locally compact quantum group (M, Δ). *We suppose throughout this section that (M, Δ) is strongly regular.* We fix a closed quantum subgroup (M_1, Δ_1). So, we have $\hat{\pi} : \hat{M}_1 \to \hat{M}$.

Suppose that $\eta : C \to \mathcal{M}(A_1 \otimes C)$ is a continuous coaction of (A_1, Δ_1) on a C*-algebra C. We want to define an induced C*-algebra $\mathrm{Ind}\, C$ with a continuous coaction $\mathrm{Ind}\, \eta$ of (A, Δ) on $\mathrm{Ind}\, C$. Of course, when $C = \mathbb{C}$ with the trivial coaction, we want to find again D with the coaction Δ of (A, Δ) by left translations on D.

We defined D as a suitable C*-subalgebra of $Q = M^\alpha$, where $\alpha : M \to M \otimes M_1$ is the coaction of (M_1, Δ_1) on M by right translations. To define C, we have again at our disposal a C*-algebra which is too big and inside which we want to find $\mathrm{Ind}\, C$.

Notation 7.1. We denote

$$\widetilde{C} = \{X \in \mathcal{M}(\mathcal{K}(H) \otimes C) \mid X \in (M' \otimes 1)' \quad \text{and} \quad (\alpha \otimes \iota)(X) = (\iota \otimes \eta)(X)\} \ .$$

We equip $\widetilde{C}$ with the strict topology inherited from $\mathcal{M}(\mathcal{K}(H) \otimes C)$ and call this the strict topology of $\widetilde{C}$.

Remark that the expression $X \in \mathcal{M}(\mathcal{K}(H) \otimes C) \cap (M' \otimes 1)'$ is the necessarily awkward way of saying that $X \in M \otimes \mathcal{M}(C)$ in some loose sense. Observe that, when $C = \mathbb{C}$, we have $\widetilde{C} = M^\alpha = Q$.

Theorem 7.2. *There exists a unique C*-subalgebra $\mathrm{Ind}\, C$ of $\widetilde{C}$ that satisfies the following conditions.*

- $\Delta \otimes \iota : \mathrm{Ind}\, C \to \mathcal{M}(A \otimes \mathrm{Ind}\, C)$ *defines a continuous coaction of (A, Δ) on $\mathrm{Ind}\, C$.*

- $\Delta \otimes \iota : \widetilde{C} \to \mathcal{M}(\mathcal{K}(H) \otimes \mathrm{Ind}\, C)$ *is well defined and strictly continuous on the unit ball of $\widetilde{C}$.*

- $\mathrm{Ind}\, C \subset \widetilde{C}$ *is non-degenerate in the sense that $H \otimes C = [(\mathrm{Ind}\, C)(H \otimes C)]$.*

We define $\mathrm{Ind}\, \eta := \Delta \otimes \iota$ and call it the induced coaction of η.

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The following theorem shows that our definition of $\operatorname{Ind} C$ is the correct one.

Theorem 7.3. *There exist canonical covariant Morita equivalences*

$$\hat{A}^{\mathrm{u}} \, {}_{f}\ltimes \operatorname{Ind} C \underset{\mathrm{Morita}}{\sim} \hat{A}^{\mathrm{u}}_1 \, {}_{f}\ltimes C \quad and \quad \hat{A} \, {}_{r}\ltimes \operatorname{Ind} C \underset{\mathrm{Morita}}{\sim} \hat{A}_1 \, {}_{r}\ltimes C \ .$$

The covariance is understood with respect to the dual coactions on the crossed products.

The rest of this section will consist in proving both theorems. We start by performing again the induction procedure as in Section 4, but taking into account systematically a C*-algebra C on which is coacted by (A_1, Δ_1).

Fix a coaction $\eta : C \to \mathcal{M}(A_1 \otimes C)$ of (A_1, Δ_1) on a C*-algebra C. Let $\mathcal{E}$ be a C*-B-module and let (X, θ) be a covariant pair for η consisting of a corepresentation $X \in \mathcal{L}(A_1 \otimes \mathcal{E})$ and a representation $\theta : C \to \mathcal{L}(\mathcal{E})$. Let $\mathcal{F} = \operatorname{Ind} \mathcal{E}$ be the induced C*-B-module and let $Y = \operatorname{Ind} X$ be the induced corepresentation of (A, Δ) on $\mathcal{F}$. We claim that there exists a canonical strict *-homomorphism $\widetilde{\theta} : \widetilde{C} \to \mathcal{L}(\mathcal{F})$ which is covariant in the sense that

$$(\iota \otimes \widetilde{\theta})(\Delta \otimes \iota)(z) = Y^*(1 \otimes \widetilde{\theta}(z))Y \quad \text{for all } z \in \widetilde{\theta} \ . \tag{7.1}$$

In order to give a meaning to the previous equality, we have to be careful. We consider

$$\widetilde{C}_1 = \{x \in \mathcal{M}(\mathcal{K}(H) \otimes \mathcal{K}(H) \otimes C) \mid x \in (1 \otimes M' \otimes 1)' \quad \text{and} \quad (\iota \otimes \alpha \otimes \iota)(x) = (\iota \otimes \iota \otimes \eta)(x)\} \ .$$

The algebra $\widetilde{C}_1$ plays the role of $\mathrm{B}(H) \otimes \widetilde{C}$. It is not difficult to define $\iota \otimes \widetilde{\theta}$ as a strict *-homomorphism $\widetilde{C}_1 \to \mathcal{L}(H \otimes \mathcal{F})$ (see Lemma 12.12 for a related result) . On the other hand, we have $\Delta \otimes \iota : \widetilde{C} \to \widetilde{C}_1$. As a composition of both, the left hand side of (7.1) makes sense.

In the induction procedure for corepresentations, an important role is played by the W*-imprimitivity bimodule $\mathcal{I}$ defined in Definition 4.2. We extend $\mathcal{I}$ as follows.

Notation 7.4. We define

$$\mathcal{J} = \{x \in \mathcal{L}(H_1 \otimes C, H \otimes C) \mid (\hat{\pi}' \otimes \iota)(V_1)_{12} x_{13} V_{1,12}^* = (\iota \otimes \eta)(x)\} \ .$$

Observe that $\mathcal{I} \otimes 1 \subset \mathcal{J}$. We also define

$$P_1 = \{x \in \mathcal{M}(\mathcal{K}(H_1) \otimes C) \mid V_{1,12} x_{13} V_{1,12}^* = (\iota \otimes \eta)(x)\}$$

and

$$P = \{x \in \mathcal{M}(\mathcal{K}(H) \otimes C) \mid V_{12} x_{13} V_{12}^* = (\iota \otimes \eta)(x)\} \ .$$

Then, $\mathcal{J}$ is a P-P_1-bimodule, $\mathcal{J}^* \mathcal{J} \subset P_1$ and $\mathcal{J} \mathcal{J}^* \subset P$. Observe also that $\widetilde{C} = P \cap (M' \otimes 1)'$.

Recall that the induced C*-B-module $\mathcal{F}$ is defined by $H \otimes \mathcal{F} \cong \mathcal{I} \underset{\pi_\ell}{\otimes} (H \otimes \mathcal{E})$, where $\pi_\ell : \hat{M}_1 \to \mathcal{L}(H \otimes \mathcal{E})$ is the strict *-homomorphism defined by $(\iota \otimes \pi_\ell)(W_1) = (\iota \otimes \hat{\pi})(W_1)_{12} X_{13}$.

We extend π_ℓ to P_1 as follows. Let $z \in P_1$. It is easy to check that

$$V_{1,12} \left(X(\iota \otimes \theta)(z)X^* \right)_{13} V_{1,12}^* = \left(X(\iota \otimes \theta)(z)X^* \right)_{13} \ .$$

So, $X(\iota \otimes \theta)(z)X^* \in \mathcal{L}(H_1 \otimes \mathcal{E}) \cap (\hat{M}'_1 \otimes 1)'$. Hence, we expect to be able to apply $\hat{\pi} \otimes \iota$ to $X(\iota \otimes \theta)(z)X^*$. Exactly as in the definition of π_ℓ we are a little bit more careful and we define the strict *-homomorphism

$$\widetilde{\pi}_\ell : P_1 \to \mathcal{L}(H \otimes \mathcal{E}) : \widetilde{\pi}_\ell(z)(v \otimes 1)\xi = (v \otimes 1) \, X(\iota \otimes \theta)(z)X^* \, \xi$$

for all $z \in P_1, \xi \in H_1 \otimes \mathcal{E}$ and $v \in \mathrm{B}(H_1, H)$ intertwining $\hat{\pi}$. So, we can define the C*-B-module $\mathcal{J} \underset{\widetilde{\pi}_\ell}{\otimes} (H \otimes \mathcal{E})$ and the following B-linear inclusion that preserves inner products.

$$\mathcal{I} \underset{\pi_\ell}{\otimes} (H \otimes \mathcal{E}) \hookrightarrow \mathcal{J} \underset{\widetilde{\pi}_\ell}{\otimes} (H \otimes \mathcal{E}) \ .$$

We claim that this inclusion is unitary. So, we have to prove that the image of this inclusion is dense. For this it suffices to show that $(\mathcal{I} \otimes 1)P_1$ is strictly dense in $\mathcal{J}$. But, $(\mathcal{I}^* \otimes 1)\mathcal{J} \subset P_1$, which implies that $(\mathcal{I} \otimes 1)P_1$ contains $(\mathcal{I}\mathcal{I}^* \otimes 1)\mathcal{J}$. Now, $\mathcal{I}\mathcal{I}^*$ is weakly dense in $\hat{M} \ltimes Q$ and in particular, 1 can be approximated by elements in $\mathcal{I}\mathcal{I}^*$ and we are done.

As a conclusion, we have
$$H \otimes \mathcal{F} \cong \mathcal{I} \underset{\pi_\ell}{\otimes} (H \otimes \mathcal{E}) = \mathcal{J} \underset{\widetilde{\pi}_\ell}{\otimes} (H \otimes \mathcal{E}) \ .$$

This shows that we have a natural strict *-homomorphism $P \to \mathcal{L}(H \otimes \mathcal{F})$.

Although P is not a von Neumann algebra, we have defined, in a sense, a B-correspondence
$$_P\boxed{H \otimes \mathcal{F}}_{\hat{M}} \ .$$

Finally, we have to turn this B-correspondence into a bicovariant B-correspondence. In Section 4, we already defined the product coaction $\alpha_{\widetilde{\mathcal{F}}}$ on $\widetilde{\mathcal{F}} = H \otimes \mathcal{F}$. We want to show that this coaction is covariant with respect to the representation $P \to \mathcal{L}(H \otimes \mathcal{F})$ together with the right coaction $z \mapsto \hat{V}_{13}(z \otimes 1)\hat{V}_{13}^*$ of $(\hat{A}, \hat{\Delta})$ on P.

For this, we have to observe that
$$z \mapsto \hat{V}_{13}(z \otimes 1)(\iota \otimes \hat{\pi})(\hat{V}_1^*)_{13} \ , \quad z \in \mathcal{J}$$

defines a right coaction of $(\hat{A}, \hat{\Delta})$ on $\mathcal{J}$. Then, the inclusion $\mathcal{I} \otimes 1 \subset \mathcal{J}$ is compatible with the coaction $\alpha_{\mathcal{I}}$ defined on $\mathcal{I}$ in Section 4. Hence, the coaction $\alpha_{\widetilde{\mathcal{F}}}$ can be considered as a product coaction on $\mathcal{J} \underset{\widetilde{\pi}_\ell}{\otimes} (H \otimes \mathcal{E})$.

Then, it is clear that we get a bicovariant B-correspondence
$$_P\overset{M'}{\boxed{H \otimes \mathcal{F}}}_{\hat{M}} \ .$$

Since $\widetilde{C} \subset P$ and since $\hat{V}_{13}(z \otimes 1)\hat{V}_{13}^* = z \otimes 1$ for all $z \in \widetilde{C}$, we have found a strict *-homomorphism
$$\widetilde{C} \to \mathcal{L}(\mathcal{F})$$

which is covariant with respect to the induced corepresentation Y.

By now it should be clear that the following lemma can be proved in the same way as Theorem 5.1.

Lemma 7.5. *Let (A_1, Δ_1) be a closed quantum subgroup of a strongly regular locally compact quantum group (A, Δ). Let $\eta : C \to \mathcal{M}(A_1 \otimes C)$ be a continuous coaction of (A_1, Δ_1) on a C^*-algebra C.*

A corepresentation Y of (A, Δ) on a C^*-B-module $\mathcal{F}$ is induced from a covariant pair (X, θ) consisting of a corepresentation X of (A_1, Δ_1) and a representation of C if and only if there exists a strict *-homomorphism $\widetilde{C} \to \mathcal{L}(F)$ which is covariant with respect to Y.

A final ingredient to prove Theorems 7.2 and 7.3 is Proposition 6.6. Let $\beta : \mathcal{E} \to \mathcal{M}(\mathcal{E} \otimes \hat{A})$ be a coaction of $(\hat{A}, \hat{\Delta})$ on $\mathcal{E}$ which is compatible with a continuous coaction on B still denoted by β. Suppose that β and the corepresentation $X \in \mathcal{L}(A_1 \otimes \mathcal{E})$ are covariant in the sense of (6.1). Then, Proposition 6.6 yields an induced coaction $\mathrm{Ind}\,\beta$ on $\mathcal{F}$. Also the following lemma can be proved easily.

Lemma 7.6. *If $\theta(C)$ is part of the fixed point algebra of $\mathcal{L}(\mathcal{E})$ with respect to β, then $\widetilde{C}$ is part of the fixed point algebra of $\mathcal{L}(\mathcal{F})$ with respect to $\mathrm{Ind}\,\beta$.*

We have then gathered enough material to prove Theorems 7.2 and 7.3.

Proof of Theorem 7.2. Consider the C^*-$\hat{A}_1 \,_r\!\ltimes C$-module $\hat{A}_1 \,_r\!\ltimes C$ together with the reduced covariant pair (X, θ) consisting of the corepresentation $X = W_{1,12} \in \mathcal{L}(A_1 \otimes (\hat{A}_1 \,_r\!\ltimes C))$ and the representation $\theta = \eta : C \to \mathcal{M}(\hat{A}_1 \,_r\!\ltimes C)$.

Define $\mathcal{F} = \mathrm{Ind}(\hat{A}_1 \,_r\!\ltimes C)$, together with the induced corepresentation $Y \in \mathcal{L}(A \otimes \mathcal{F})$ and the strict *-homomorphism $\widetilde{C} \to \mathcal{L}(\mathcal{F})$.

On the crossed product $\hat{A}_1 \ {}_r\!\ltimes C$, we have the dual coaction that we push into $(\hat{A}, \hat{\Delta})$ using $\hat{\pi}$:

$$\hat{\eta} : \hat{A}_1 \ {}_r\!\ltimes C \to \mathcal{M}\big((\hat{A}_1 \ {}_r\!\ltimes C) \otimes \hat{A}\big) \ .$$

By Lemma 7.6, we find a coaction γ on $\mathcal{F}$ such that $(\iota \otimes \gamma)(Y) = W_{13}Y_{12}$ and such that $\widetilde{C}$ is part of the fixed point algebra $\mathcal{L}(\mathcal{F})^{\gamma}$.

By Theorem 6.7 and Remark 6.8, we find a C*-algebra $\operatorname{Ind} C \subset \mathcal{L}(\mathcal{F})^{\gamma}$ such that

- $z \mapsto Y^*(1 \otimes z)Y$ defines a continuous coaction of (A, Δ) on $\operatorname{Ind} C$;

- $\mathcal{L}(\mathcal{F})^{\gamma} \to \mathcal{M}(\mathcal{K}(H) \otimes \operatorname{Ind} C) : z \mapsto Y^*(1 \otimes z)Y$ is well defined and strictly continuous;

- $\mathcal{K}(\mathcal{F}) = [(\operatorname{Ind} C) \, (\omega \otimes \iota)(Y) \mid \omega \in \mathrm{B}(H)_*]$.

Consider then the regular representation $\hat{A}_1 \ {}_r\!\ltimes C \to \mathcal{L}(H_1 \otimes C)$. It is straightforward to check that $\operatorname{Ind}(H_1 \otimes C) = H \otimes C$. Since

$$(\operatorname{Ind}(\hat{A}_1 \ {}_r\!\ltimes C)) \underset{\hat{A}_1 \ {}_r\!\ltimes C}{\otimes} (H_1 \otimes C) = \operatorname{Ind}(H_1 \otimes C) = H \otimes C \ ,$$

we find a faithful *-homomorphism $\mu : \mathcal{L}(\mathcal{F}) \to \mathcal{L}(H \otimes C)$.

One verifies that $\mu(\mathcal{L}(\mathcal{F})) \subset P$. Further, μ intertwines the coaction γ on $\mathcal{L}(\mathcal{F})$ with the coaction $z \mapsto \hat{V}_{13}(z \otimes 1)\hat{V}_{13}^*$ of $(\hat{A}, \hat{\Delta})$ on P. Hence, $\mu\big(\mathcal{L}(\mathcal{F})^{\gamma}\big) \subset \widetilde{C}$. So, we have shown that $\mathcal{L}(\mathcal{F})^{\gamma} \cong \widetilde{C}$ in a canonical way and that the isomorphism preserves the strict topology on bounded subsets.

As such, we have defined $\operatorname{Ind} C$ in a canonical way as a subalgebra of $\widetilde{C}$ such that $\operatorname{Ind} C$ satisfies the required conditions. The uniqueness of $\operatorname{Ind} C$ is proved in exactly the same way as the uniqueness statement in Remark 6.8. $\qquad\square$

We finally prove Theorem 7.3.

Proof of Theorem 7.3. In the proof of Theorem 7.2 the reduced Morita equivalence has already been shown. We shall prove the full Morita equivalence.

We can almost copy the proof of Theorem 6.2. The only point to show is the following: let $Y \in \mathcal{L}(A \otimes \mathcal{F})$ be a corepresentation of (A, Δ) on a C*-B-module $\mathcal{F}$ which is covariant with respect to a representation $\rho : \operatorname{Ind} C \to \mathcal{L}(\mathcal{F})$, then ρ extends uniquely to a strict *-homomorphism $\widetilde{C} \to \mathcal{L}(\mathcal{F})$ which remains covariant. Once we have proved this claim, we can use Lemma 7.5.

The uniqueness of the extension is obvious from the requirement of covariance and from the existence of the map $\Delta \otimes \iota : \widetilde{C} \to \mathcal{M}(\mathcal{K}(H) \otimes \operatorname{Ind} C)$.

To prove the existence of the extension of ρ, we define

$$\mu : \widetilde{C} \to \mathcal{L}(H \otimes \mathcal{F}) : \mu(z) = Y(\iota \otimes \rho)(\Delta \otimes \iota)(z)Y^* \ .$$

Observe that $\mu(z) = 1 \otimes \rho(z)$ for all $z \in \operatorname{Ind} C$. Let $z \in \widetilde{C}$. Clearly, $\mu(z) \in (M' \otimes 1)'$. Moreover,

$$(\Delta \otimes \iota)\mu(z) = Y_{13}Y_{23} \, (\iota \otimes \iota \otimes \rho)(\iota \otimes \Delta \otimes \iota)(\Delta \otimes \iota)(z) \, Y_{23}^*Y_{13}^* \ .$$

Since $(\Delta \otimes \iota)(z) \in \mathcal{M}(\mathcal{K}(H) \otimes \operatorname{Ind} C)$, the covariance of ρ implies that

$$Y_{23} \, (\iota \otimes \iota \otimes \rho)(\iota \otimes \Delta \otimes \iota)(\Delta \otimes \iota)(z) \, Y_{23}^* = \big((\iota \otimes \rho)(\Delta \otimes \iota)(z)\big)_{13} \ .$$

It follows that $(\Delta \otimes \iota)\mu(z) = \mu(z)_{13}$, which implies that $\mu(z) = 1 \otimes \rho_1(z)$ for a well defined strict *-homomorphism $\rho_1 : \widetilde{C} \to \mathcal{L}(\mathcal{F})$. Then, ρ_1 is the required extension. $\qquad\square$

Remark 7.7. In order to define $\operatorname{Ind} C$, we did not assume that the coaction of (A_1, Δ_1) on C is reduced. One checks easily that the induction of (C, η_1) coincides with the induction of the reduction of (C, η_1). This is not very surprising, since our induced coaction is reduced by definition. Nevertheless, we prefer not to impose that η_1 is reduced, to get neater formulations in the next section.

8   Induction and restriction

We still fix a strongly regular locally compact quantum group (M, Δ) with a closed quantum subgroup (M_1, Δ_1). Suppose that $\eta : C \to \mathcal{M}(A \otimes C)$ is a continuous coaction of (A, Δ) on the C*-algebra C. We shall deal with the following problem: what happens if we first restrict the coaction η to a coaction $\eta_1 : C \to \mathcal{M}(A_1 \otimes C)$ and then induce η_1?

In the classical situation, where G_1 is a closed subgroup of a locally compact group G and where G is acting continuously on C, we know that the induction of the restriction will be $C_0(G/G_1) \otimes C$ with the diagonal action. There is an obvious reason for which we cannot expect exactly the same result: if M is non-commutative, the notion of a diagonal coaction does not make sense in general. We shall see however in Proposition 8.2 that there is a natural description of the induction of the restriction of a coaction, where the tensor product with diagonal action is replaced by a twisted product and diagonal coaction.

Of course we should first discuss the construction of the restriction of η to η_1. Recall that the morphism $(M, \Delta) \xrightarrow{\pi} (M_1, \Delta_1)$ comes as a non-degenerate *-homomorphism $\pi : A^{\mathrm{u}} \to \mathcal{M}(A_1^{\mathrm{u}})$. Hence, the obvious formula $\eta_1 = (\pi \otimes \iota)\eta$ does not make sense immediately.

Definition 8.1. Let $\eta : C \to \mathcal{M}(A \otimes C)$ be a continuous coaction which admits a lift to a coaction $C \to \mathcal{M}(A^{\mathrm{u}} \otimes \mathbb{C})$. Then the restriction $\eta_1 : C \to \mathcal{M}(A_1 \otimes C)$ is the unique coaction satisfying

$$(\iota \otimes \eta_1)\eta = (\alpha \otimes \iota)\eta \, ,$$

where $\alpha : A \to \mathcal{M}(A \otimes A_1)$ is the right coaction of (A_1, Δ_1) on (A, Δ). The restriction η_1 is continuous.

Observe that η admits a unique lift to a coaction $C \to \mathcal{M}(A^{\mathrm{u}} \otimes C)$ when either η is reduced or maximal, but that an arbitrary continuous coaction does not necessarily admit such a lift.

Theorem 8.2. *Let $\eta : C \to \mathcal{M}(A \otimes C)$ be a continuous coaction of (A, Δ) on the C*-algebra C, admitting a lift to a coaction $C \to \mathcal{M}(A^{\mathrm{u}} \otimes C)$. Restrict η to a coaction $\eta_1 : C \to \mathcal{M}(A_1 \otimes C)$. The induced C*-algebra for η_1 is given by*

$$\mathrm{Ind}(C, \eta_1) \cong [(D \otimes 1)\eta(C)]$$

with induced coaction $\Delta \otimes \iota$.

Moreover, $D \to \mathcal{M}(D \otimes \hat{A}) : x \mapsto W(x \otimes 1)W^*$ defines a continuous right coaction of $(\hat{A}, \hat{\Delta}^{\mathrm{op}})$ on D. We call this the adjoint coaction. *If η is a reduced coaction, we have*

$$\mathrm{Ind}(C, \eta_1) \cong [\, (W(D \otimes 1)W^*)_{12} \, \eta(C)_{23} \,] \, .$$

Remark that in the case where M is commutative, the adjoint coaction is trivial and we find that $\mathrm{Ind}(C, \eta_1) \cong D \otimes C$.

In order to prove Theorem 8.2, we have to do some preliminary work. We have shown in Theorem 6.2 that $\hat{A} \underset{r}{\ltimes} D \underset{\mathrm{Morita}}{\sim} \hat{A}_1$. In fact, the Morita equivalence is given by $\mathcal{F} = \mathrm{Ind} \, \hat{A}_1$, which means that

$$H \otimes \mathcal{F} \cong \mathcal{I} \underset{\pi_\ell}{\otimes} (H \otimes \hat{A}_1) \, .$$

We have seen that $\mathcal{I} \underset{\pi_\ell}{\otimes} (H \otimes \hat{A}_1)$ is a bicovariant $\hat{A}_1$-correspondence:

$$\hat{M} \ltimes Q \boxed{ \begin{array}{c} M' \\ \mathcal{I} \underset{\pi_\ell}{\otimes} (H \otimes \hat{A}_1) \end{array} } \hat{M} \, .$$

Consider now the W*-$\hat{M}_1$-module $H \otimes \mathcal{I}$. Then, equipped with the obvious representations of $\hat{M} \ltimes Q$, $\hat{M}' \otimes 1$, $M' \otimes 1$, we get an inclusion

$$\mathcal{I} \underset{\pi_\ell}{\otimes} (H \otimes \hat{A}_1) \hookrightarrow H \otimes \mathcal{I} : v \underset{\pi_\ell}{\otimes} \xi \mapsto \hat{V}_{21}(1 \otimes v)(\hat{\pi} \otimes \iota)(\hat{V}_{1,21}^*)\xi$$

intertwining these three representations. It follows that there is a closed subspace $\mathcal{I}_0 \subset \mathcal{I}$ such that (applying a flip map)

- $\mathcal{I}_0 \otimes H = [\hat{V}(\mathcal{I} \otimes 1)(\iota \otimes \hat{\pi})(\hat{V}_1^*)(\hat{A}_1 \otimes H)]$;

- $[\mathcal{I}_0^* \mathcal{I}_0] = \hat{A}_1$;

- $[\mathcal{I}_0 \mathcal{I}_0^*] = [\hat{A}D]$;

- $v \mapsto \hat{V}(v \otimes 1)(\iota \otimes \hat{\pi})(\hat{V}_1^*)$ defines a continuous coaction of $(\hat{A}, \hat{\Delta})$ on $\mathcal{I}_0$.

We should consider $\mathcal{I}_0$ as a *concrete realization of the $\hat{A} \,_r\!\ltimes D$-$\hat{A}_1$ imprimitivity bimodule*.

Suppose now that $\eta_1 : C \to \mathcal{M}(A_1 \otimes C)$ is a continuous coaction of (A_1, Δ_1) on the C*-algebra C. We can give an analogous concrete $\hat{A} \,_r\!\ltimes \operatorname{Ind} C$ - $\hat{A}_1 \,_r\!\ltimes C$ imprimitivity bimodule. From the proof of Theorem 7.2, we know that such an imprimitivity bimodule is given by $\operatorname{Ind}(\hat{A}_1 \,_r\!\ltimes C)$. In exactly the same way as above, we then find that

$$\operatorname{Ind}(\hat{A}_1 \,_r\!\ltimes C) \cong [(\mathcal{I}_0 \otimes 1)\eta_1(C)] \subset \mathcal{M}(\mathcal{K}(H_1, H) \otimes C)$$

as a covariant C*-bimodule.

Proof of Theorem 8.2. We first prove the claim on the adjoint coaction. On the W*-$\hat{M}_1$-module $\mathcal{I}$ we consider the coaction $\mathcal{I} \to \mathcal{I} \otimes \hat{M} : v \mapsto W(v \otimes 1)(\iota \otimes \hat{\pi})(W_1)^*$ of $(\hat{M}, \hat{\Delta}^{\mathrm{op}})$ on the right, which is compatible with the coaction $(\iota \otimes \hat{\pi})\hat{\Delta}_1^{\mathrm{op}} : \hat{M}_1 \to \hat{M}_1 \otimes \hat{M}$ of $(\hat{M}, \hat{\Delta}^{\mathrm{op}})$ on $\hat{M}_1$. We also consider the coaction $\iota \otimes (\iota \otimes \hat{\pi})\hat{\Delta}_1^{\mathrm{op}}$ of $(\hat{A}, \hat{\Delta}^{\mathrm{op}})$ on $H \otimes \hat{A}_1$. From Proposition 12.13 we get a product coaction of $(\hat{A}, \hat{\Delta}^{\mathrm{op}})$ on $\mathcal{I} \underset{\pi_\ell}{\otimes} (H \otimes \hat{A}_1)$. Since it is clear that this product coaction leaves invariant the representations of M' and $\hat{M}'$, we get a coaction

$$\gamma : \operatorname{Ind} \hat{A}_1 \to \mathcal{M}((\operatorname{Ind} \hat{A}_1) \otimes \hat{A})$$

of $(\hat{A}, \hat{\Delta}^{\mathrm{op}})$ on the imprimitivity bimodule $\operatorname{Ind} \hat{A}_1$, which is compatible with the coaction $(\iota \otimes \hat{\pi})\hat{\Delta}_1^{\mathrm{op}}$ of $(\hat{A}, \hat{\Delta}^{\mathrm{op}})$ on $\hat{A}_1$. If we follow the isomorphism $\operatorname{Ind} \hat{A}_1 \cong \mathcal{I}_0$, the coaction γ is given by $\gamma : \mathcal{I}_0 \mapsto \mathcal{M}(\mathcal{I}_0 \otimes \hat{A}) : v \mapsto W(v \otimes 1)(\iota \otimes \hat{\pi})(W_1^*)$.

In particular, we get a coaction $\gamma : \hat{A} \,_r\!\ltimes D \to \mathcal{M}((\hat{A} \,_r\!\ltimes D) \otimes \hat{A})$. By construction, the coaction γ commutes with the dual coaction on $\hat{A} \,_r\!\ltimes D$ and is given by $\hat{\Delta}^{\mathrm{op}}$ on $\hat{A}$. This implies (see the proof of Theorem 6.7) that the restriction of γ to D is a continuous coaction of $(\hat{A}, \hat{\Delta}^{\mathrm{op}})$ on D. By construction, this coaction is exactly given by $x \mapsto W(x \otimes 1)W^*$.

Above we have concretely realized the $\hat{A} \,_r\!\ltimes \operatorname{Ind} C$ - $\hat{A}_1 \,_r\!\ltimes C$ imprimitivity bimodule as $[(\mathcal{I}_0 \otimes 1)\eta_1(C)]$. We claim that $[(\mathcal{I}_0 \otimes 1)\eta_1(C)] = [\eta(C)(\mathcal{I}_0 \otimes 1)]$. To prove this claim, we write $\widetilde{\eta}(x) = (J\hat{J} \otimes 1)\eta(x)(\hat{J}J \otimes 1)$ and observe that

$$(\iota \otimes \eta_1)\widetilde{\eta}(x) = (\hat{\pi} \otimes \iota)(\hat{W}_1^*)_{12}\widetilde{\eta}(x)_{13}(\hat{\pi} \otimes \iota)(\hat{W}_1)_{12} \quad \text{and} \quad (\iota \otimes \eta)\widetilde{\eta}(x) = \hat{W}_{12}^*\widetilde{\eta}(x)_{13}\hat{W}_{12} \ .$$

But then, by continuity of the coaction $\gamma : \mathcal{I}_0 \to \mathcal{M}(\mathcal{I}_0 \otimes \hat{A})$ defined above, we get

$$
\begin{aligned}
[\mathcal{K} \otimes (\mathcal{I}_0 \otimes 1)\eta_1(C)] &= [(\mathcal{K} \otimes \mathcal{I}_0 \otimes 1)(\iota \otimes \eta_1)\widetilde{\eta}(C)] \\
&= [((1 \otimes \mathcal{K})\gamma(\mathcal{I}_0) \otimes 1)(\iota \otimes \hat{\pi})(W_1)_{12}\widetilde{\eta}(C)_{23}(\iota \otimes \hat{\pi})(W_1^*)_{12}]_{213} \\
&= [(1 \otimes \mathcal{K} \otimes 1)W_{12}(\mathcal{I}_0 \otimes \widetilde{\eta}(C))(\iota \otimes \hat{\pi})(W_1^*)_{12}]_{213} \\
&= [(\mathcal{K} \otimes 1 \otimes 1)(\iota \otimes \eta)\widetilde{\eta}(C)\gamma(\mathcal{I}_0)_{21}] \\
&= [(\mathcal{K} \otimes \eta(C))\gamma(\mathcal{I}_0)_{21}] = \mathcal{K} \otimes [\eta(C)(\mathcal{I}_0 \otimes 1)] \ .
\end{aligned}
$$

This proves our claim. Since $\hat{A} \,_r\!\ltimes \operatorname{Ind} C \cong \mathcal{K}(\operatorname{Ind}(\hat{A}_1 \,_r\!\ltimes C))$, we conclude that

$$\hat{A} \,_r\!\ltimes \operatorname{Ind} C \cong [(\mathcal{I}_0 \otimes 1)\eta_1(C)(\mathcal{I}_0^* \otimes 1)\eta(C)] = [(\mathcal{I}_0 \mathcal{I}_0^* \otimes 1)\eta(C)] = [(\hat{A}D \otimes 1)\eta(C)]$$

in such a way that the dual coaction on $\hat{A} \,_r\!\ltimes \operatorname{Ind} C$ corresponds with the coaction $z \mapsto \hat{V}_{13}(z \otimes 1)\hat{V}_{13}^*$ of $(\hat{A}, \hat{\Delta})$ on $[(\hat{A}D \otimes 1)\eta(C)]$. It follows from Theorem 6.7 that $\operatorname{Ind} C \cong [(D \otimes 1)\eta(C)]$ in such a way that the induced coaction $\operatorname{Ind} \eta_1$ corresponds to $\Delta \otimes \iota$.

We finally observe that, when η is a reduced coaction

$$[(W(D \otimes 1)W^*)_{12}\eta(C)_{23}] = W_{12}(\iota \otimes \eta)([(D \otimes 1)\eta(C)])W_{12}^* \cong [(D \otimes 1)\eta(C)] \ .$$

This ends the proof of the theorem. $\qquad\square$

We give a slightly more natural explanation why $[(D \otimes 1)\eta(C)]$ is indeed a C*-algebra. It is an example of a kind of *twisted product* generalizing reduced crossed products.

Proposition 8.3. *Let (A, Δ) be a locally compact quantum group and let C, D be C*-algebras. Suppose that $\gamma : D \to \mathcal{M}(\hat{A} \otimes D)$ is a continuous coaction of $(\hat{A}, \hat{\Delta})$ on D and that $\alpha : C \to \mathcal{M}(A \otimes C)$ is a continuous coaction of (A, Δ) on C. Then,*

$$[\gamma(D)_{12} \, \alpha(C)_{13}] \subset \mathcal{M}(\mathcal{K}(H) \otimes D \otimes C)$$

is a C-algebra.*

This proposition indeed generalizes the notion of reduced crossed product. If $\alpha : C \to \mathcal{M}(A \otimes C)$ is a continuous coaction, we take $D = \hat{A}$ with the coaction $\gamma = \hat{\Delta}$. We observe that, using the notation σ for the flip,

$$(\sigma \otimes \iota)[\hat{\Delta}(\hat{A})_{12} \, \alpha(C)_{13}] = W_{12}[(\hat{A} \otimes 1 \otimes 1)(\iota \otimes \alpha)\alpha(C)]W_{12}^* = (WV)_{12}(\hat{A} \,_r\!\ltimes C)_{13}(WV)_{12}^* \cong \hat{A} \,_r\!\ltimes C .$$

To obtain $\mathrm{Ind}(C, \eta_1)$ as an example of such a twisted product, we take the adjoint coaction on the quantum homogeneous space D and the given coaction $\eta : C \to \mathcal{M}(A \otimes C)$ on C.

Proof. By the continuity of γ, we get

$$\gamma(D) = [(\omega \otimes \iota \otimes \iota)(\hat{\Delta} \otimes \iota)\gamma(D)] = [(\omega \otimes \iota \otimes \iota)(\hat{V}_{12}\gamma(D)_{13}\hat{V}_{12}^*)] = [(\omega \otimes \iota \otimes \iota)(W_{12}^*\tilde{\gamma}(D)_{13}W_{12})] ,$$

where $\tilde{\gamma}(x) = (J\hat{J} \otimes 1)\gamma(x)(\hat{J}J \otimes 1)$. Using the continuity of α, it follows that

$$\begin{aligned}
[\gamma(D)_{12} \, \alpha(C)_{13}] &= [(\omega \otimes \iota \otimes \iota \otimes \iota)(W_{12}^*\tilde{\gamma}(D)_{13}W_{12} \, (\mathcal{K}(H) \otimes \alpha(C))_{124})] \\
&= [(\omega \otimes \iota \otimes \iota \otimes \iota)(W_{12}^*\tilde{\gamma}(D)_{13}W_{12} \, (\Delta \otimes \iota)\alpha(C)_{124})] \\
&= [(\omega \otimes \iota \otimes \iota \otimes \iota)(W_{12}^*\tilde{\gamma}(D)_{13} \, \alpha(C)_{24}W_{12})] \\
&= [(\omega \otimes \iota \otimes \iota \otimes \iota)((\Delta \otimes \iota)\alpha(C)_{124} \, W_{12}^*\tilde{\gamma}(D)_{13}W_{12})] \\
&= [(\omega \otimes \iota \otimes \iota \otimes \iota)(\alpha(C)_{24} \, W_{12}^*\tilde{\gamma}(D)_{13}W_{12})] = [\alpha(C)_{13} \, \gamma(D)_{12}] .
\end{aligned}$$

It is then clear that $[\gamma(D)_{12} \, \alpha(C)_{13}]$ is a C*-algebra. $\qquad\square$

As an application of these twisted products, we define *reduced crossed products by homogeneous spaces*.

Proposition 8.4. *Let (A, Δ) be a strongly regular locally compact quantum group with a closed quantum subgroup (A_1, Δ_1). Denote the quantum homogeneous space by D.*

Let $\mu : C \to \mathcal{M}(\hat{A} \otimes C)$ be continuous coaction of $(\hat{A}, \hat{\Delta})$ on a C-algebra C. Then,*

$$[(D \otimes 1)\mu(C)]$$

is a C-algebra that we denote by $D \,_r\!\ltimes C$ and that we call the* reduced crossed product *of C by D.*

Proof. It suffices to observe that $\Delta : D \to \mathcal{M}(A \otimes D)$ is a continuous coaction of (A, Δ) on D. By Proposition 8.3 we get that $[\Delta(D)_{12}\mu(C)_{13}]$ is a C*-algebra. But,

$$[\Delta(D)_{12}\mu(C)_{13}] = W_{12}^*(\sigma \otimes \iota)[(D \otimes 1 \otimes 1)(\iota \otimes \mu)\mu(C)]W_{12} = W_{12}^*\hat{V}_{21}(1 \otimes [(D \otimes 1)\mu(C)])\hat{V}_{21}^*W_{12} .$$

It follows that $[(D \otimes 1)\mu(C)]$ is a C*-algebra. $\qquad\square$

Finally it is possible to define as well *full crossed products by homogeneous spaces*. Nevertheless we should remark that our definition does not fully generalize full crossed products by quantum groups (see Remark 8.6 below).

Definition 8.5. Let (A, Δ) be a strongly regular locally compact quantum group with a closed quantum subgroup (A_1, Δ_1). Denote the quantum homogeneous space by D. Let $\mu : C \to \mathcal{M}(\hat{A} \otimes C)$ be continuous coaction of $(\hat{A}, \hat{\Delta})$ on a C*-algebra C.

A pair (π_D, π_C) of non-degenerate representations of D and C on a Hilbert space K is said to be *covariant* if the C*-algebras

$$(\iota \otimes \pi_D)(V(1 \otimes D)V^*) \quad \text{and} \quad (\iota \otimes \pi_C)\mu(C)$$

commute as C*-algebras on $H \otimes K$.

If (π_D, π_C) is such a covariant representation, then $[\pi_D(D)\pi_C(C)]$ is a C*-algebra. The C*-algebra generated by a universal covariant representation is denoted by $D \rtimes_f C$ and called the *full crossed product of C by D*.

Observe that the expression $(\iota \otimes \pi_D)(V(1 \otimes D)V^*)$ makes sense because the adjoint coaction is a well defined continuous coaction on D.

Remark 8.6. Suppose that, in the setting of the previous definition, $A_1 = \mathbb{C}$, the one-point subgroup. Then, of course, $D = A$. One should expect that $D \rtimes_f C$ coincides with the usual full crossed product. This is not necessarily the case: the representation π_D is a representation of $D = A$ and not of the full C*-algebra A^{u}.

It is possible to define a *universal homogeneous space* and to use this one, rather than D, to define full crossed products. We do not go into this: in the situation where we use the full crossed product by D (Theorem 10.1), the coaction μ is a dual coaction and hence, there is no need to take a universal variant of D.

9 Characterization of induced coactions

It is well known that a continuous action of a l.c. group G on a C*-algebra B is induced from an action of a closed subgroup G_1 if and only if there exists a G-equivariant embedding of $C_0(G/G_1)$ into the center of $\mathcal{M}(B)$. We prove a similar result in this section. Nevertheless, we cannot hope for an identical characterization since it would require the commutativity of the quantum homogeneous space D.

We fix a strongly regular l.c. quantum group (M, Δ) together with a closed quantum subgroup (M_1, Δ_1) given by the morphism $\hat{\pi} : \hat{M}_1 \to \hat{M}$. We denote as above the quantum homogeneous space by D, which is a C*-subalgebra of the measured quantum homogeneous space $Q = M^\alpha$, where α is the right coaction of (M_1, Δ_1) on M by right translations.

Lemma 9.1. *There is a canonical anti-automorphism γ of $\hat{A} \rtimes_r D$ given by $\gamma(z) = Jz^*J$ when $\hat{A} \rtimes_r D$ is represented on H.*

Proof. Consider the concrete $\hat{A} \rtimes_r D$ - $\hat{A}_1$ imprimitivity bimodule $\mathcal{I}_0 \subset \mathrm{B}(H_1, H)$ constructed in the previous section. We claim that $\mathcal{I}_0 = J\mathcal{I}_0 J_1$. It is clear that the lemma follows from this claim. First of all, $J\mathcal{I}J_1 = \mathcal{I}$, since $J\hat{\pi}(x)J = \hat{R}(\hat{\pi}(x^*)) = \hat{\pi}(\hat{R}_1(x^*)) = \hat{\pi}(J_1 x J_1)$ for all $x \in \hat{M}_1$. Hence, it follows that

$$J\mathcal{I}_0 J_1 \otimes H = [W(J\mathcal{I}J_1 \otimes 1)(\iota \otimes \hat{\pi})(W_1^*)(\hat{A}_1 \otimes H)] = [W(\mathcal{I} \otimes 1)(\iota \otimes \hat{\pi})(W_1^*)(\hat{A}_1 \otimes H)]$$
$$\supset [W(\mathcal{I}_0 \otimes 1)(\iota \otimes \hat{\pi})(W_1^*)(\hat{A}_1 \otimes H)] .$$

In the proof of Theorem 8.2, we have shown that the coaction $v \mapsto W(v \otimes 1)(\iota \otimes \hat{\pi})(W_1^*)$ is continuous. It then follows that $J\mathcal{I}_0 J_1 \otimes H \supset \mathcal{I}_0 \otimes H$, which proves our claim. $\qquad\square$

Theorem 9.2. *Let $\eta : B \to \mathcal{M}(A \otimes B)$ be a continuous reduced coaction of (A, Δ) on a C*-algebra B. Then η is induced from a coaction of (A_1, Δ_1) if and only if there exists a non-degenerate *-homomorphism $\theta : D \to \mathcal{M}(B)$ which is covariant and satisfies the condition*

$$\widetilde{\theta}(\gamma(D)) \quad commutes \ with \quad B \quad in \quad \hat{A} \rtimes_r B \ ,$$

where $\widetilde{\theta} : \hat{A} \rtimes_r D \to \mathcal{M}(\hat{A} \rtimes_r B)$ denotes the extension of θ and γ is the anti-automorphism defined in Lemma 9.1.

Observe that, in the case where $M = L^\infty(G)$ is abelian, we have $\gamma(D) = D$ and the above condition exactly says that $\theta(D)$ is in the center of $\mathcal{M}(B)$.

Proof. Consider again the concrete $\hat{A}\ _r\!\ltimes D$ - $\hat{A}_1$ imprimitivity bimodule $\mathcal{I}_0 \subset \mathrm{B}(H_1, H)$ constructed in the previous section. If η_1 is a continuous coaction of (A_1, Δ_1) on C, we can realize $\mathrm{Ind}\, C \subset \mathcal{M}(A \otimes C)$ such that
$$\hat{A}\ _r\!\ltimes \mathrm{Ind}\, C = [(\hat{A} \otimes 1)\,\mathrm{Ind}\, C] = [(\mathcal{I}_0 \otimes 1)\eta_1(C)(\mathcal{I}_0^* \otimes 1)] \ .$$

It then follows easily that we can embed $\theta : D \to \mathcal{M}(\mathrm{Ind}\, C) : \theta(x) = x \otimes 1$ such that $\widetilde{\theta}(\gamma(D)) = JDJ \otimes 1 \subset M' \otimes 1 \subset (\mathrm{Ind}\, C)'$.

Suppose conversely that we have a continuous coaction of (A, Δ) on B and a *-homomorphism $\theta : D \to \mathcal{M}(B)$ with the properties stated in the theorem. Define $\mathcal{E} = \mathcal{I}_0^* \underset{\widetilde{\theta}}{\otimes} (\hat{A}\ _r\!\ltimes B)$ and denote by ρ the coaction of $(\hat{A}, \hat{\Delta})$ on $\mathcal{E}$ on the right, which is the internal tensor product of the coaction $\mathcal{I}_0^* \to \mathcal{M}(\mathcal{I}_0^* \otimes \hat{A}) : v \to (\iota \otimes \hat{\pi})(\hat{V}_1)(v \otimes 1)\hat{V}^*$ and the dual coaction on $\hat{A}\ _r\!\ltimes B$. We still write ρ for the corresponding coaction on $\mathcal{K}(\mathcal{E})$. Observe that we have a representation $\hat{A}_1 \to \mathcal{L}(\mathcal{E})$ which is covariant with respect to the coaction $(\iota \otimes \hat{\pi}_1)\hat{\Delta}_1$ of $(\hat{A}, \hat{\Delta})$ on $\hat{A}_1$ and the coaction ρ on $\mathcal{L}(\mathcal{E})$.

We write $F := \mathcal{K}(\mathcal{E})$. We claim that $\rho : F \to \mathcal{M}(F \otimes \hat{A})$ is of the form $(\iota \otimes \hat{\pi})\rho_1$ for a continuous coaction ρ_1 of $(\hat{A}_1, \hat{\Delta}_1)$ on F. It then follows from Theorem 6.7 that $F = \hat{A}_1\ _r\!\ltimes C$ for some continuous coaction η_1 of (A_1, Δ_1) on C. By construction we have $(B, \eta) \cong (\mathrm{Ind}\, C, \mathrm{Ind}\, \eta_1)$.

To prove our claim, take $x \in D$, $v \in \mathcal{I}_0$ and $b \in B$. Then

$$(1 \otimes \hat{J}x\hat{J})\rho(v^* \underset{\widetilde{\theta}}{\otimes} b) = (1 \otimes \hat{J}x\hat{J})\big((\iota \otimes \hat{\pi})(\hat{V}_1)(v^* \otimes 1)\hat{V}^* \underset{\widetilde{\theta}\otimes\iota}{\otimes} (b \otimes 1)\big)$$

$$= \big((\iota \otimes \hat{\pi})(\hat{V}_1)(v^* \otimes 1)\hat{V}^* (J \otimes \hat{J})\Delta^{\mathrm{op}}(x)(J \otimes \hat{J})\big) \underset{\widetilde{\theta}\otimes\iota}{\otimes} (b \otimes 1)$$

$$= \big((\iota \otimes \hat{\pi})(\hat{V}_1)(v^* \otimes 1)\hat{V}^* (\gamma \otimes R)\Delta^{\mathrm{op}}(x^*)\big) \underset{\widetilde{\theta}\otimes\iota}{\otimes} (b \otimes 1)$$

$$= (\iota \otimes \hat{\pi})(\hat{V}_1)(v^* \otimes 1)\hat{V}^* \underset{\widetilde{\theta}\otimes\iota}{\otimes} (\widetilde{\theta}\gamma \otimes R)\Delta^{\mathrm{op}}(x^*)(b \otimes 1) = \rho(v^* \underset{\widetilde{\theta}}{\otimes} b)\, (\widetilde{\theta}\gamma \otimes R)\Delta^{\mathrm{op}}(x^*) \ .$$

It follows from the calculation that $1 \otimes \hat{J}D\hat{J}$ and $\rho(F)$ commute. Since $M \cap (\hat{J}D\hat{J})' = \hat{\pi}(\hat{M}_1)$, it follows that $\rho(F)$ has its second leg in $\hat{\pi}(\hat{M}_1)$. So, we can define $\rho_1 : F \to \mathcal{M}(F \otimes \mathcal{K}(H_1)) \cap (1 \otimes \hat{M}_1')'$ such that $(1 \otimes w)\rho_1(x) = \rho(x)(1 \otimes w)$ whenever $x \in F$ and $w \in \mathrm{B}(H_1, H)$ with $wy = \hat{\pi}(y)w$ for all $y \in \hat{M}_1$. It follows that $[\rho_1(F)(1 \otimes H_1)] = F \otimes H_1$. Since ρ is a continuous coaction, we find that

$$[\rho_1(F)(1 \otimes \hat{A}_1) \otimes H_1] = [(\rho_1 \otimes \iota)\rho_1(F)(1 \otimes \hat{A}_1 \otimes H_1)] = [\hat{W}_{1,23}^* \rho_1(F)_{13}\hat{W}_{1,23}(1 \otimes \hat{A}_1 \otimes H_1)]$$

$$= [\hat{W}_{1,23}^* \rho_1(F)_{13}(1 \otimes \hat{A}_1 \otimes H_1)] = [\hat{W}_{1,23}^*(F \otimes \hat{A}_1 \otimes H_1)] = F \otimes \hat{A}_1 \otimes H_1 \ .$$

This proves that ρ_1 is a continuous coaction of $(\hat{A}_1, \hat{\Delta}_1)$ on F and hence, proves our claim and the theorem. $\square$

10 Green-Rieffel-Mansfield imprimitivity

We still fix a strongly regular locally compact quantum group (M, Δ) with a closed quantum subgroup (M_1, Δ_1). So, we have $\hat{\pi} : \hat{M}_1 \to \hat{M}$.

Suppose that $\eta : C \to \mathcal{M}(JAJ \otimes C)$ is a continuous coaction of (JAJ, Δ') on a C*-algebra C, which admits a lift to the universal level. (It is not crucial to take a left coaction of JAJ rather then a right coaction of (A, Δ). The only convenience is that the crossed product is now $\hat{A} \ltimes C$ with the dual action being a left coaction of $(\hat{A}, \hat{\Delta})$ such that the second crossed product is $A \ltimes \hat{A} \ltimes C$.) The comultiplication Δ' is defined by

$$\Delta'(JxJ) = (J \otimes J)\Delta(x)(J \otimes J)$$

for all $x \in A$.

We then have a restricted coaction $\eta_1 : C \to \mathcal{M}(J_1 A_1 J_1 \otimes C)$.

Theorem 10.1. *If η is a reduced coaction, there is a canonical Morita equivalence*

$$D \ {}_r\!\ltimes \hat{A} \ {}_r\!\ltimes C \underset{\text{Morita}}{\sim} \hat{A}_1 \ {}_r\!\ltimes C \ .$$

If η is a maximal coaction, there is a canonical Morita equivalence

$$D \ {}_f\!\ltimes \hat{A}^{\mathrm{u}} \ {}_f\!\ltimes C \underset{\text{Morita}}{\sim} \hat{A}_1^{\mathrm{u}} \ {}_f\!\ltimes C \ .$$

The conditions for η being reduced or maximal are very natural. Indeed, if (M_1, Δ_1) is the one-point subgroup of (M, Δ), a natural Morita equivalence between the second crossed product $A \ {}_r\!\ltimes \hat{A} \ {}_r\!\ltimes C$ and C exists exactly when η is reduced. On the other hand, a natural Morita equivalence between the second crossed product $A^{\mathrm{u}} \ {}_f\!\ltimes \hat{A}^{\mathrm{u}} \ {}_f\!\ltimes C$ and C exists exactly when η is maximal.

The reduced and full crossed products by the quantum homogeneous space D have been defined in Proposition 8.4 and Definition 8.5.

Observe that $D \ {}_f\!\ltimes \hat{A}^{\mathrm{u}} \ {}_f\!\ltimes C$ is the universal C*-algebra defined by covariant triples (ρ, Y, θ) consisting of commuting representations ρ of D and θ of C, and a corepresentation $Y \in \mathcal{M}(A \otimes \mathcal{K}(K))$ satisfying the covariance relations

$$(\iota \otimes \rho)\Delta(x) = Y^*(1 \otimes \rho(x))Y \quad \text{for all } x \in D \quad \text{and} \quad (\iota \otimes \theta)\widetilde{\eta}(y) = Y(1 \otimes \theta(y))Y^* \quad \text{for all } y \in C \ . \quad (10.1)$$

Here we used the following notation

$$\widetilde{\eta} : C \to \mathcal{M}(A \otimes C) : \widetilde{\eta}(y) = (J\hat{J} \otimes 1)\eta(y)(\hat{J}J \otimes 1) \ .$$

For later use, we also introduce the notation

$$\widetilde{\eta}_1 : C \to \mathcal{M}(A_1 \otimes C) : \widetilde{\eta}_1(y) = (J_1 \hat{J}_1 \otimes 1)\eta_1(y)(\hat{J}_1 J_1 \otimes 1) \ .$$

Then, $\widetilde{\eta}$ is a coaction of $(A, \Delta^{\mathrm{op}})$ on C and $\widetilde{\eta}_1$ is its restriction to $(A_1, \Delta_1^{\mathrm{op}})$.

Proof. Let (X, θ_1) be a covariant representation for the coaction η_1 on the C*-B-module $\mathcal{E}$, consisting of a corepresentation $X \in \mathcal{L}(A_1 \otimes \mathcal{E})$ and a representation $\theta_1 : C \to \mathcal{L}(\mathcal{E})$ satisfying the relation

$$(\iota \otimes \theta_1)\widetilde{\eta}(y) = X(1 \otimes \theta_1(y))X^* \quad \text{for all } y \in C \ .$$

We induce the corepresentation X to a corepresentation $Y := \mathrm{Ind}\, X$ on the induced C*-B-module $\mathcal{F} := \mathrm{Ind}\, \mathcal{E}$. Recall that

$$H \otimes \mathcal{F} \cong \mathcal{I} \underset{\pi_\ell}{\otimes} (H \otimes \mathcal{E}) \ , \quad (10.2)$$

where the representation $\pi_\ell : \hat{M}_1 \to \mathcal{L}(H \otimes \mathcal{E})$ is determined by $(\iota \otimes \pi_\ell)(W_1) = (\iota \otimes \hat{\pi})(W_1)_{12} X_{13}$. Using the covariance of θ_1, it is easily checked that $\pi_\ell(\hat{M}_1)$ and $(\iota \otimes \theta_1)\widetilde{\eta}(C)$ commute. So, we get a well defined representation

$$\widetilde{\theta} : C \to \mathcal{L}(H \otimes \mathcal{F}) : \widetilde{\theta}(y)(v \underset{\pi_\ell}{\otimes} \xi) = v \underset{\pi_\ell}{\otimes} (\iota \otimes \theta_1)\widetilde{\eta}(y)\xi \quad \text{for all } v \in \mathcal{I}, \xi \in H \otimes \mathcal{E} \ .$$

We implicitly used the identification in (10.2) when defining $\widetilde{\theta}$. Since $(\iota \otimes \theta_1)\widetilde{\eta}(C)$ commutes with $M' \otimes 1$, it follows that $\widetilde{\theta}(C)$ commutes with $M' \otimes 1$ as well. By definition of $\widetilde{\theta}$, we have

$$(\iota \otimes \widetilde{\theta})\widetilde{\eta}(y) = V_{21}(1 \otimes \widetilde{\theta}(y))V_{21}^* \quad \text{for all } y \in C \ .$$

Finally, we have $\pi_\ell : \hat{M} \ltimes Q \to \mathcal{L}(H \otimes \mathcal{F})$ and the images of $\pi_\ell(\hat{M} \ltimes Q)$ and $\widetilde{\theta}(C)$ commute. This means in particular that $1 \otimes \widetilde{\theta}(C)$ commutes with $W_{12} Y_{13}$.

It follows that, for all $y \in C$,

$$W_{12}^*(Y^*\widetilde{\theta}(y)Y)_{23}W_{12} = Y_{23}^*Y_{13}^*W_{12}^*(\iota \otimes \widetilde{\theta}(y))W_{12}Y_{13}Y_{23} = (Y^*\widetilde{\theta}(y)Y)_{23} \ .$$

Hence, $Y^*\widetilde{\theta}(y)Y$ commutes with $\hat{M} \otimes 1$. We already know that $Y^*\widetilde{\theta}(y)Y$ commutes with $M' \otimes 1$. So, we find a representation $\theta : C \to \mathcal{L}(\mathcal{F})$ such that $\widetilde{\theta}(y) = Y(1 \otimes \theta(y))Y^*$. From the relations stated above, we conclude that the image of θ commutes with the image of the homogeneous space $\rho : D \to \mathcal{L}(\mathcal{F})$ and that $(\iota \otimes \theta)\widetilde{\eta}(y) = Y(1 \otimes \theta(y))Y^*$ for all $y \in C$.

So, we have found a triple (ρ, Y, θ), which means a representation of $D \ {}_f\ltimes \hat{A}^{\mathrm{u}} \ {}_f\ltimes C$ on $\mathcal{F} = \mathrm{Ind}\,\mathcal{E}$.

We can perform as well the inverse induction. Let $\mathcal{F}$ be a C*-B-module. Suppose that (ρ, Y, θ) is a triple consisting of commuting representations ρ of D and θ of C on $\mathcal{F}$, and a corepresentation $Y \in \mathcal{L}(A \otimes \mathcal{F})$ satisfying the covariance relations in (10.1).

As in the proof of Theorem 5.1, we find the C*-B-module $\mathcal{E}$ by $H \otimes \mathcal{E} \cong \mathcal{I}^* \underset{\pi_\ell}{\otimes} (H \otimes \mathcal{F})$, where $\pi_\ell : \hat{M} \ltimes Q \to$ $\mathcal{L}(H \otimes \mathcal{F})$ is the strict *-homomorphism determined by

$$(\iota \otimes \pi_\ell)(W) = W_{12}Y_{13} \quad \text{and} \quad \pi_\ell(x) = 1 \otimes \rho(x) \quad \text{for all } x \in D \ .$$

We find a corepresentation $X \in \mathcal{L}(A_1 \otimes \mathcal{E})$ such that $Y = \mathrm{Ind}\,X$.

Defining $\widetilde{\theta} : C \to \mathcal{L}(H \otimes \mathcal{F}) : \widetilde{\theta}(y) = Y(1 \otimes \theta(y))Y^*$, it is clear that the images of $\widetilde{\theta}$ and π_ℓ commute. So, we obtain a representation $\widetilde{\theta}_1 : C \to \mathcal{L}(H \otimes \mathcal{E})$ given by

$$\widetilde{\theta}_1(y)(v \underset{\pi_\ell}{\otimes} \xi) = v \underset{\pi_\ell}{\otimes} \widetilde{\theta}(y)\xi \quad \text{for all } v \in \mathcal{I}^*, \xi \in H \otimes \mathcal{F} \ .$$

Exactly as it was the case in the induction procedure above, we find that $\widetilde{\theta}_1(C)$ commutes with $M' \otimes 1$ and satisfies

$$V_{21}(1 \otimes \widetilde{\theta}_1(y))V_{21}^* = (\iota \otimes \widetilde{\theta}_1)\widetilde{\eta}(y) \quad \text{for all } y \in C \ . \tag{10.3}$$

Finally, $1 \otimes \widetilde{\theta}_1(C)$ commutes with $(\iota \otimes \hat{\pi})(W_1)_{12}X_{13}$.

Suppose now first that η is a maximal coaction. This means that the natural surjective *-homomorphism $A^{\mathrm{u}} \ {}_f\ltimes \hat{A}^{\mathrm{u}} \ {}_f\ltimes C \to \mathcal{K}(H) \otimes C$ is an isomorphism. At the end of the previous paragraph, we found a representation $\widetilde{\theta}_1$ of C on $H \otimes \mathcal{E}$, which commutes with $M' \otimes 1$ and satisfies the covariance relation (10.3) with respect to $\hat{M}' \otimes 1$. The maximality of η implies that there exists a unique representation $\theta_1 : C \to \mathcal{L}(\mathcal{E})$ such that $\widetilde{\theta}_1 = (\iota \otimes \theta_1)\widetilde{\eta}$. It follows easily that θ_1 is covariant with respect to X. So, we have found a covariant pair (X, θ_1) such that (ρ, Y, θ) is the induction of (X, θ_1). In the same way as we have shown Theorem 6.2, it follows that there exists a canonical Morita equivalence

$$D \ {}_f\ltimes \hat{A}^{\mathrm{u}} \ {}_f\ltimes C \underset{\mathrm{Morita}}{\sim} \hat{A}_1^{\mathrm{u}} \ {}_f\ltimes C \ .$$

More concretely, the Morita equivalence can be written as $\mathrm{Ind}(\hat{A}_1^{\mathrm{u}} \ {}_f\ltimes C)$.

Suppose next that η is an arbitrary continuous coaction. Consider $\hat{A}_1 \ {}_r\ltimes C$ as a C*-$\hat{A}_1 \ {}_r\ltimes C$-module and define $\mathcal{F} = \mathrm{Ind}(\hat{A}_1 \ {}_r\ltimes C)$. Exactly as before the proof of Theorem 8.2 we realize $\mathcal{F}$ concretely as

$$\mathcal{F} \cong [(\mathcal{I}_0 \otimes 1)\eta_1(C)] \subset \mathcal{L}(H_1 \otimes C, H \otimes C) \ .$$

As in the proof of Theorem 8.2, but now using the continuous coaction $\mathcal{I}_0 \to \mathcal{M}(\mathcal{I}_0 \otimes \hat{A})$ given by $v \mapsto \hat{V}(v \otimes 1)(\iota \otimes \hat{\pi})(\hat{V}_1^*)$, we find that $[(\mathcal{I}_0 \otimes 1)\eta_1(C)] = [\eta(C)(\mathcal{I}_0 \otimes 1)]$. It follows that

$$\mathcal{K}(\mathcal{F}) \cong [(\mathcal{I}_0 \otimes 1)\eta_1(C)(\mathcal{I}_0^* \otimes 1)\eta(C)] = [(\mathcal{I}_0\mathcal{I}_0^* \otimes 1)\eta(C)] = [(D\hat{A} \otimes 1)\eta(C)] \ .$$

So we found a canonical Morita equivalence $\hat{A}_1 \ {}_r\ltimes C \underset{\mathrm{Morita}}{\sim} [(D\hat{A} \otimes 1)\eta(C)]$. Suppose that η is reduced. Then

$$\hat{V}_{12} \ (\iota \otimes \eta)[(D\hat{A} \otimes 1)\eta(C)] \ \hat{V}_{12}^* = [(D \otimes 1 \otimes 1)(\hat{\Delta}(\hat{A}) \otimes 1)(1 \otimes \eta(C))] = D \ {}_r\ltimes \hat{A} \ {}_r\ltimes C$$

and we are done. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 10.2. Observe that we have shown that for arbitrary continuous coactions $\eta : C \to \mathcal{M}(JAJ \otimes C)$, admitting a lift to the universal level, there is a canonical Morita equivalence

$$\hat{A}_1 \, _r\ltimes C \underset{\text{Morita}}{\sim} [(D\hat{A} \otimes 1)\eta(C)] \ .$$

11   Final remarks

The particular case of inducing a unitary corepresentation of a closed quantum subgroup has been treated by Kustermans [14]. His approach, in the spirit of Mackey, does not allow to prove C*-algebraic imprimitivity theorems. Of course, one can verify that his induction is unitarily equivalent to ours. Nevertheless one needs to use the complete machinery of modular theory to prove this result. This is not surprising: the induced corepresentation of Kustermans involves the canonical implementation (in the sense of [26]) of the coaction of (M, Δ) on Q and this is essentially an object in modular theory. The key result that one has to prove is that the induction of the trivial corepresentation of (M_1, Δ_1) is exactly this unitary implementation.

From the naturality and functoriality of our induction procedure, it follows immediately that there is a theorem on *induction in stages*: if (M_2, Δ_2) is a closed quantum subgroup of (M_1, Δ_1) and the latter is a closed quantum subgroup of (M, Δ), then inducing first from (M_2, Δ_2) to (M_1, Δ_1) and then from (M_1, Δ_1) to (M, Δ) is the same as inducing from (M_2, Δ_2) to (M, Δ).

In the case where the closed quantum subgroup (M_1, Δ_1) of (M, Δ) is *normal*, i.e. when we have a short exact sequence $e \longrightarrow (M_2, \Delta_2) \longrightarrow (M, \Delta) \longrightarrow (M_1, \Delta_1) \longrightarrow e$, it follows immediately from the uniqueness statement in Theorem 6.1 that the quantum homogeneous space is exactly the reduced C*-algebra of the quantum group (M_2, Δ_2).

The latter example shows moreover that a quantum homogeneous space satisfying the conditions in Theorem 6.1 may exist even in the non-regular or non-semi-regular case.

12   Appendix: C*- and W*-modules and their coactions

12.1   Coactions on C*-modules

We briefly recall from [2] the notion of a coaction on a Hilbert C*-module.

Notation 12.1. Let $\mathcal{E}$ be a C*-B-module. Then we denote

$$\mathcal{M}(\mathcal{E}) = \mathcal{L}(B, \mathcal{E}) \ .$$

Definition 12.2. Let $\alpha_B : B \to \mathcal{M}(B \otimes A)$ be a coaction of (A, Δ) on B and let $\mathcal{E}$ be a C*-B-module. A coaction of (A, Δ) on $\mathcal{E}$ compatible with α_B is a linear map

$$\alpha_{\mathcal{E}} : \mathcal{E} \to \mathcal{M}(\mathcal{E} \otimes A)$$

satisfying

a) $\alpha_{\mathcal{E}}(vx) = \alpha_{\mathcal{E}}(v) \, \alpha_B(x)$ for all $v \in \mathcal{E}, x \in B$;
$\langle \alpha_{\mathcal{E}}(v), \alpha_E(w) \rangle = \alpha_B(\langle v, w \rangle)$ for all $v, w \in \mathcal{E}$;

b) the linear span of $\alpha_{\mathcal{E}}(\mathcal{E})(B \otimes A)$ is dense in $B \otimes A$;

c) $(\alpha_{\mathcal{E}} \otimes \iota)\alpha_{\mathcal{E}} = (\iota \otimes \Delta)\alpha_{\mathcal{E}}$ (which makes sense because of 1) and 2), see [2] for details).

Let $\alpha_{\mathcal{E}}$ be a coaction of (A, Δ) on $\mathcal{E}$ compatible with α_B. Then we construct a unitary operator

$$\mathcal{V} : \mathcal{E} \underset{\alpha_B}{\otimes} (B \otimes A) \to \mathcal{E} \otimes A : \mathcal{V}(v \underset{\alpha_B}{\otimes} x) = \alpha_{\mathcal{E}}(v)x \ .$$

It is easy to verify that $\mathcal{V}$ satisfies the relation

$$(\mathcal{V}\underset{\mathbb{C}}{\otimes}\iota)(\mathcal{V}\underset{\alpha_B\otimes\iota}{\otimes}1)=\mathcal{V}\underset{\iota\otimes\Delta}{\otimes}1\,. \tag{12.1}$$

This equality holds in $\mathcal{L}(\mathcal{E}\underset{(\iota\otimes\Delta)\alpha_B}{\otimes}(B\otimes A\otimes A),\mathcal{E}\otimes A\otimes A)$ and its correct interpretation uses the following identifications.

$$
\begin{array}{ccc}
\mathcal{E}\underset{(\alpha_B\otimes\iota)\alpha_B}{\otimes}(B\otimes A\otimes A) & =\!=\!=\!= & \mathcal{E}\underset{(\iota\otimes\Delta)\alpha_B}{\otimes}(B\otimes A\otimes A)\\[4pt]
\simeq\downarrow & & \downarrow\simeq\\[4pt]
\big(\mathcal{E}\underset{\alpha_B}{\otimes}(B\otimes A)\big)\underset{\alpha_B\otimes\iota}{\otimes}(B\otimes A\otimes A) & & \big(\mathcal{E}\underset{\alpha_B}{\otimes}(B\otimes A)\big)\underset{\iota\otimes\Delta}{\otimes}(B\otimes A\otimes A)\\[4pt]
\mathcal{V}\underset{\alpha_B\otimes\iota}{\otimes}1\downarrow & & \downarrow\\[4pt]
(\mathcal{E}\otimes A)\underset{\alpha_B\otimes\iota}{\otimes}(B\otimes A\otimes A) & & \downarrow\ \ \mathcal{V}\underset{\iota\otimes\Delta}{\otimes}1\\[4pt]
\simeq\downarrow & & \downarrow\\[4pt]
\big(\mathcal{E}\underset{\alpha_B}{\otimes}(B\otimes A)\big)\otimes A & & (\mathcal{E}\otimes A)\underset{\iota\otimes\Delta}{\otimes}(B\otimes A\otimes A)\\[4pt]
\mathcal{V}\underset{\mathbb{C}}{\otimes}\iota\downarrow & & \downarrow\simeq\\[4pt]
\mathcal{E}\otimes A\otimes A & =\!=\!=\!= & \mathcal{E}\otimes A\otimes A
\end{array}
\tag{12.2}
$$

Whenever we write the symbol $\simeq$ in this diagram, we mean that there is a natural identification, not involving the coaction on $\mathcal{E}$.

Proposition 12.3. *Let $\alpha_B:B\to\mathcal{M}(B\otimes A)$ be a coaction and $\mathcal{E}$ a C^*-B-module. Let $\alpha_{\mathcal{E}}:\mathcal{E}\to\mathcal{M}(\mathcal{E}\otimes A)$ be a linear map. Then, the following conditions are equivalent.*

a) $\alpha_{\mathcal{E}}$ is a coaction of (A,Δ) on $\mathcal{E}$ compatible with α_B.

b) There exists a coaction on the link algebra $\mathcal{K}(\mathcal{E}\oplus B)$ that coincides with $\alpha_{\mathcal{E}}$ on $\mathcal{E}$ and α_B on B.

c) The formula $\mathcal{V}(v\underset{\alpha_B}{\otimes}x)=\alpha_{\mathcal{E}}(v)x$ defines a unitary operator $\mathcal{V}$ in $\mathcal{L}(\mathcal{E}\underset{\alpha_B}{\otimes}(B\otimes A),\mathcal{E}\otimes A)$ satisfying (12.1).

Remark 12.4. Let $\mathcal{E}$ be a C^*-B-module. If $\mathcal{V}$ defines a coaction of (A,Δ) on $\mathcal{E}$ which is compatible with the trivial coaction on B, then $\mathcal{V}\in\mathcal{L}(\mathcal{E}\otimes A,\mathcal{E}\otimes A)=\mathcal{M}(\mathcal{K}(\mathcal{E})\otimes A)$ and as such, $\mathcal{V}$ is a corepresentation of (A,Δ) in $\mathcal{K}(\mathcal{E})$.

Remark 12.5. If (A,Δ) is a regular l.c. quantum group and α_B a continuous coaction, then the associated coaction on the link algebra is automatically continuous.

Indeed, from the compatibility of $\alpha_{\mathcal{E}}$ and α_B, as well as the continuity of α_B, it follows that $\mathcal{E}=[(\iota\otimes\omega)\alpha_{\mathcal{E}}(\mathcal{E})]$. One can repeat to prove of Proposition 5.8 in [3] to obtain that $[(1\otimes A)\alpha_{\mathcal{E}}(\mathcal{E})]=\mathcal{E}\otimes A$ and then we are done.

12.2 Coactions on W*-modules

Usually, a von Neumann algebra M is defined as a C^*-algebra acting non-degenerately on a Hilbert space such that one of the following equivalent conditions holds true: M is weakly closed, the unit ball of M is strongly-* closed, M is equal to its bicommutant M''.

We define in the same way the notion of a W*-M-module. The reader should convince himself that the proofs of the following two propositions are elementary.

Proposition 12.6. *Let $M \subset \mathrm{B}(H)$ be a von Neumann algebra and let $\mathcal{E}$ be a C^*-M-module. Then, the following conditions are equivalent.*

a) *$\mathcal{E} \underset{M}{\otimes} 1 \subset \mathrm{B}(H, \mathcal{E} \underset{M}{\otimes} H)$ is weakly closed.*

b) *The unit ball of $\mathcal{E} \underset{M}{\otimes} 1$ is strongly-* closed.*

c) *$\mathcal{E} \underset{M}{\otimes} 1 = \{T \in \mathrm{B}(H, \mathcal{E} \underset{M}{\otimes} H) \mid Tx = (1 \underset{M}{\otimes} x)T \quad \text{for all } x \in M'\}$.*

d) *The link algebra $\mathcal{L}(\mathcal{E} \oplus M) = \begin{pmatrix} \mathcal{L}(\mathcal{E}) \underset{M}{\otimes} 1 & \mathcal{E} \underset{M}{\otimes} 1 \\ (\mathcal{E} \underset{M}{\otimes} 1)^* & M \end{pmatrix} \subset \mathrm{B}\big((\mathcal{E} \underset{M}{\otimes} H) \oplus H\big)$ is a von Neumann algebra.*

If one of these conditions holds true, we call $\mathcal{E}$ a W^*-M-module.

In the following way, we extend the notion of a normal, unital *-homomorphism to W^*-modules.

Proposition 12.7. *Let $M \subset \mathrm{B}(H_M)$ and $N \subset \mathrm{B}(H_N)$ be von Neumann algebras and $\pi_M : M \to N$ a normal, unital *-homomorphism. Let $\mathcal{E}$ be a W^*-M-module and $\mathcal{F}$ a W^*-N-module.*

Suppose that $\pi_{\mathcal{E}} : \mathcal{E} \to \mathcal{F}$ is a linear map such that

- *$\pi_{\mathcal{E}}(vx) = \pi_{\mathcal{E}}(v)\pi_M(x) \quad \text{for all } v \in \mathcal{E}, x \in M$,*

- *$\langle \pi_{\mathcal{E}}(v), \pi_{\mathcal{E}}(w) \rangle = \pi_M(\langle v, w \rangle) \quad \text{for all } v, w \in \mathcal{E}$.*

Then, $\pi_{\mathcal{E}}$ is automatically strongly continuous on the unit ball of $\mathcal{E}$. Moreover, the following conditions are equivalent.*

a) *$\pi_{\mathcal{E}}(\mathcal{E})H_N := (\pi_{\mathcal{E}}(\mathcal{E}) \underset{N}{\otimes} 1)H_N$ is dense in $\mathcal{F} \underset{N}{\otimes} H_N$.*

b) *$\pi_{\mathcal{E}}$ and π_M extend to a unital, normal, *-homomorphism $\mathcal{L}(\mathcal{E} \oplus M) \to \mathcal{L}(\mathcal{F} \oplus N)$.*

If one of these conditions holds true, we say that $\pi_{\mathcal{E}}$ is a non-degenerate morphism compatible with π_M. In that case, the extension to the link algebra $\mathcal{L}(\mathcal{E} \oplus M)$ is unique.

Having spelled out the notion of a non-degenerate morphism, we can study coactions on W^*-modules.

Remark that is obvious how to define outer and interior tensor products of W^*-modules, in the same spirit as for C^*-modules.

Definition 12.8. Let $\alpha_N : N \to N \otimes M$ be a coaction of a l.c. quantum group (M, Δ) on the von Neumann algebra N. Let $\mathcal{E}$ be a W^*-N-module.

Let $\alpha_{\mathcal{E}} : \mathcal{E} \to \mathcal{E} \otimes M$ be a non-degenerate morphism compatible with α_N. Then, the following two conditions are equivalent.

a) $(\alpha_{\mathcal{E}} \otimes \iota)\alpha_{\mathcal{E}} = (\iota \otimes \Delta)\alpha_{\mathcal{E}}$.

b) The extension of $\alpha_{\mathcal{E}}$ and α_N to the link algebra $\mathcal{L}(\mathcal{E} \oplus N)$ is a coaction of (M, Δ) on the von Neumann algebra $\mathcal{L}(\mathcal{E} \oplus N)$.

In that case, we say that $\alpha_{\mathcal{E}}$ is a coaction of (M, Δ) on $\mathcal{E}$ compatible with α_N.

12.3 The interior tensor product of a W*- and a C*-module

The subtle point of this paper is the construction of an interior tensor product of a W*-module and a C*-module, as well as an interior tensor product of compatible coactions. We have seen in Definition 3.1 the notion of a strict homomorphism $N \to \mathcal{L}(\mathcal{E})$, where N is a von Neumann algebra and $\mathcal{E}$ a C*-module.

The two natural constructions to develop next are the following. They are the crucial technical ingredients for the approach to induction presented in this paper.

a) Define the interior tensor product $\mathcal{I} \underset{\pi}{\otimes} \mathcal{E}$ of the W*-N-module $\mathcal{I}$ and the C*-B-module $\mathcal{E}$ when $\pi : N \to \mathcal{L}(\mathcal{E})$ is strict.

b) Given a coaction of (M, Δ) on $\mathcal{I}$ and a coaction of its C*-algebraic version (A, Δ) on $\mathcal{E}$ such that π is covariant, construct an interior tensor product coaction of (A, Δ) on $\mathcal{I} \underset{\pi}{\otimes} \mathcal{E}$ in the spirit of Baaj & Skandalis [2], who deal with the interior tensor product of coactions on C*-modules.

Definition 12.9. Let $\mathcal{I}$ be a W*-N-module and $\mathcal{E}$ a C*-B-module. Let $\pi : N \to \mathcal{L}(\mathcal{E})$ be a strict *-homomorphism.

Then, the algebraic tensor product $\mathcal{I} \underset{\pi}{\overset{\mathrm{alg}}{\otimes}} \mathcal{E}$ can be completed to a C*-B-module $\mathcal{I} \underset{\pi}{\otimes} \mathcal{E}$ using the inner product

$$\langle v \underset{\pi}{\otimes} w, v' \underset{\pi}{\otimes} w' \rangle = \langle w, \pi(\langle v, v' \rangle) w' \rangle \quad \text{for} \ \ v, v' \in \mathcal{I}, w, w' \in \mathcal{E} \ .$$

Remark 12.10. Observe that when (v_i) is a bounded net in $\mathcal{I}$ converging strongly* to $v \in \mathcal{I}$ and $w \in \mathcal{E}$, then $(v_i \underset{\pi}{\otimes} w)$ converges (in norm) to $v \underset{\pi}{\otimes} w$. In particular, if $\mathcal{I}_0$ is a subspace of $\mathcal{I}$ whose unit ball is strongly* dense in $\mathcal{I}$, then the algebraic tensor product $\mathcal{I}_0 \underset{\pi}{\overset{\mathrm{alg}}{\otimes}} \mathcal{E}$ is dense in $\mathcal{I} \underset{\pi}{\otimes} \mathcal{E}$.

In the situation of Definition 12.9, $\mathcal{L}(\mathcal{I})$ is a von Neumann algebra and the *-homomorphism $\mathcal{L}(I) \to \mathcal{L}(\mathcal{I} \underset{\pi}{\otimes} \mathcal{E}) : x \mapsto x \underset{\pi}{\otimes} 1$ is a strict *-homomorphism.

Suppose now that a l.c. quantum group (M, Δ) is coacting on N by $\alpha_N : N \to N \otimes M$. Suppose that its C*-algebraic companion (A, Δ) is coacting on a C*-B-module $\mathcal{E}$. In particular, we have the coaction

$$\beta_{\mathcal{L}(\mathcal{E})} : \mathcal{L}(\mathcal{E}) \to \mathcal{L}(\mathcal{E} \otimes A) \ .$$

Suppose that $\pi : N \to \mathcal{L}(E)$ is a strict *-homomorphism. We then want to give a meaning to the covariance relation

$$(\pi \otimes \iota)\alpha_N = \beta_{\mathcal{L}(\mathcal{E})}\pi \ .$$

Using the representation of A on the Hilbert space H, we have $\mathcal{L}(\mathcal{E} \otimes A) \hookrightarrow \mathcal{L}(\mathcal{E} \otimes H)$. Lemma 12.12 below tells how to define a strict *-homomorphism $\pi \otimes \iota : M \otimes \mathrm{B}(H) \to \mathcal{L}(\mathcal{E} \otimes H)$. This leads to the following definition.

Definition 12.11. In the situation described in the previous paragraph, we say that π is covariant when the equation $(\pi \otimes \iota)\alpha_N(x) = \beta_{\mathcal{L}(\mathcal{E})}\pi(x)$ holds in $\mathcal{L}(\mathcal{E} \otimes H)$ for all $x \in N$.

Lemma 12.12. *Let N be a von Neumann algebra and $\mathcal{E}$ a C*-B-module. Suppose that $\pi : N \to \mathcal{L}(E)$ is a strict *-homomorphism. Let H be a Hilbert space. Then, there exists a unique strict *-homomorphism $\pi \otimes \iota : N \otimes \mathrm{B}(H) \to \mathcal{L}(\mathcal{E} \otimes H)$ satisfying $(\pi \otimes \iota)(x \otimes y) = \pi(x) \otimes y$.*

Proof. Consider the W*-N-module $N \otimes H$. Identifying $\mathcal{E} \otimes H \simeq (N \otimes H) \underset{\pi}{\otimes} \mathcal{E}$, we get a strict *-homomorphism $\mathcal{L}(N \otimes H) \to \mathcal{L}(\mathcal{E} \otimes H)$. It is clear that $\mathcal{L}(N \otimes H) = N \otimes \mathrm{B}(H)$ and so, we are done. $\square$

We finally want to construct the interior tensor product of a coaction on a W*-module and a coaction on a C*-module, following Baaj & Skandalis [2] who made the interior tensor product of coactions on C*-modules.

We fix the following data.

- Let $\alpha_N : N \to N \otimes M$ be a coaction of a l.c. quantum group (M, Δ) on a von Neumann algebra N. Let $\mathcal{I}$ be a W*-N-module and $\alpha_{\mathcal{I}} : \mathcal{I} \to \mathcal{I} \otimes M$ a compatible coaction of (M, Δ) on N.

- Let (A, Δ) be the C*-algebraic companion of (M, Δ). Let $\beta_B : B \to \mathcal{M}(B \otimes A)$ be a coaction of (A, Δ) on the C*-algebra B. Let $\mathcal{E}$ be a C*-B-module equipped with a compatible coaction $\beta_{\mathcal{E}} : \mathcal{E} \to \mathcal{M}(\mathcal{E} \otimes A)$.

- Let $\pi : N \to \mathcal{L}(\mathcal{E})$ be a strict *-homomorphism which is covariant in the sense of Definition 12.11.

Proposition 12.13. *In the situation above, there exists a unique coaction $\gamma_{\mathcal{F}}$ of (A, Δ) on $\mathcal{F} := \mathcal{I} \underset{\pi}{\otimes} \mathcal{E}$ compatible with β_B and satisfying*

$$\gamma_{\mathcal{F}}(v \underset{\pi}{\otimes} w)x = \alpha_{\mathcal{I}}(v) \underset{\pi \otimes \iota}{\otimes} \big(\beta_{\mathcal{E}}(w)x\big) \quad \text{for all } v \in \mathcal{I}, w \in \mathcal{E}, x \in B \otimes \mathcal{K}(H) . \tag{12.3}$$

Here we use the embeddings $\mathcal{M}(\mathcal{F} \otimes A) \hookrightarrow \mathcal{M}(\mathcal{F} \otimes \mathcal{K}(H))$ and $\mathcal{M}(\mathcal{E} \otimes A) \hookrightarrow \mathcal{M}(\mathcal{E} \otimes \mathcal{K}(H))$ to give a meaning to the previous equality.

Remark that is included in the contents of the proposition that $\gamma_{\mathcal{F}} : \mathcal{F} \to \mathcal{M}(\mathcal{F} \otimes A)$, which is non-obvious from the defining relation (12.3).

Proof. We shall write $\mathcal{K}$ for $\mathcal{K}(H)$ throughout the proof. We define a unitary

$$\mathcal{V} \in \mathcal{L}((\mathcal{I} \underset{\pi}{\otimes} \mathcal{E}) \underset{\beta_B}{\otimes} (B \otimes \mathcal{K}), (\mathcal{I} \underset{\pi}{\otimes} \mathcal{E}) \otimes \mathcal{K})$$

by the formula

$$\mathcal{V}\big((v \underset{\pi}{\otimes} w) \underset{\beta_B}{\otimes} x\big) = \alpha_{\mathcal{I}}(v) \underset{\pi \otimes \iota}{\otimes} \big(\beta_{\mathcal{E}}(w)x\big) .$$

The slightly non-trivial point to check is the surjectivity of $\mathcal{V}$. To prove this, it suffices to check that any element of $\mathcal{I} \otimes \mathrm{B}(H)$ can be approximated in the strong* topology by a bounded net in $\mathrm{span}\big(\alpha_{\mathcal{I}}(\mathcal{I})(1 \otimes \mathcal{K})\big)$. This last result follows from the fact that α_I and α_N combine to a coaction of (M, Δ) on the link algebra, on which we can apply the results of [26].

We claim that $\mathcal{V}$ satisfies the relation

$$(\mathcal{V} \underset{\mathbb{C}}{\otimes} \iota)(\mathcal{V} \underset{\beta_B \otimes \iota}{\otimes} 1) = \mathcal{V} \underset{\iota \otimes \Delta}{\otimes} 1 \tag{12.4}$$

which holds in $\mathcal{L}((\mathcal{I} \underset{\pi}{\otimes} \mathcal{E}) \underset{(\beta_B \otimes \iota)\beta_B}{\otimes} (B \otimes \mathcal{K} \otimes \mathcal{K}), (\mathcal{I} \underset{\pi}{\otimes} \mathcal{E}) \otimes \mathcal{K} \otimes \mathcal{K})$ and which should be given a precise meaning as

follows. Using the multiplicative unitary $W \in \mathcal{M}(A \otimes \hat{A})$, we define $\Delta : \mathcal{K} \to \mathcal{M}(\mathcal{K} \otimes \mathcal{K}) : \Delta(k) = W^*(1 \otimes k)W$, extending the comultiplication Δ on A. Then Equation (12.4) gets a precise meaning as in Equation (12.2) replacing systematically A by $\mathcal{K}$.

Let (e_i) be an approximate unit in $B \otimes \mathcal{K}$. Then, for $v \in \mathcal{I}$, $w \in \mathcal{E}$ and $x \in B \otimes \mathcal{K} \otimes \mathcal{K}$, we have

$$(\mathcal{V} \underset{\beta_B \otimes \iota}{\otimes} 1)\big((v \underset{\pi}{\otimes} w) \underset{(\beta_B \otimes \iota)\beta_B}{\otimes} x\big) = \lim_i (\mathcal{V} \underset{\beta_B \otimes \iota}{\otimes} 1)\big(((v \underset{\pi}{\otimes} w) \underset{\beta_B}{\otimes} e_i) \underset{\beta_B \otimes \iota}{\otimes} x\big) = \lim_i \big(\alpha_I(v) \underset{\pi \otimes \iota}{\otimes} \beta_{\mathcal{E}}(w)e_i\big) \underset{\beta_B \otimes \iota}{\otimes} x .$$

We now consider

$$(\mathcal{I} \otimes M) \underset{\pi \otimes \iota}{\otimes} (\mathcal{E} \otimes \mathcal{K}) \underset{\beta_B \otimes \iota}{\otimes} (B \otimes \mathcal{K} \otimes \mathcal{K}) \simeq \big((\mathcal{I} \underset{\pi}{\otimes} \mathcal{E}) \underset{\beta_B}{\otimes} (B \otimes \mathcal{K})\big) \otimes \mathcal{K}$$

$$\overset{\mathcal{V} \otimes \iota}{\underset{\mathbb{C}}{\to}} (\mathcal{I} \underset{\pi}{\otimes} \mathcal{E}) \otimes \mathcal{K} \otimes \mathcal{K} \simeq (\mathcal{I} \otimes M \otimes M) \underset{\pi \otimes \iota \otimes \iota}{\otimes} (B \otimes \mathcal{K} \otimes \mathcal{K}) .$$

This chain of maps applied to an elementary tensor yields

$$(v \otimes a) \underset{\pi \otimes \iota}{\otimes} (w \otimes k) \underset{\beta_B \otimes \iota}{\otimes} (y \otimes l)$$

$$\mapsto \big((v \underset{\pi}{\otimes} w) \underset{\beta_B}{\otimes} y\big) \otimes akl$$

$$\mapsto \big(\alpha_{\mathcal{I}}(v) \underset{\pi \otimes \iota}{\otimes} \beta_{\mathcal{E}}(w)y\big) \otimes akl$$

$$\mapsto (\alpha_{\mathcal{I}} \otimes \iota)(v \otimes a) \underset{\pi \otimes \iota \otimes \iota}{\otimes} \big((\beta_{\mathcal{E}} \otimes \iota)(w \otimes k)(y \otimes l)\big) .$$

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It follows that

$$(\mathcal{V} \underset{\mathbb{C}}{\otimes} \iota)(\mathcal{V} \underset{\beta_B \otimes \iota}{\otimes} 1)\big((v \otimes w) \underset{\pi}{\underset{(\beta_B \otimes \iota)\beta_B}{\otimes}} x\big) = \lim_i (\alpha_I \otimes \iota)\alpha_I(v) \underset{\pi \otimes \iota \otimes \iota}{\otimes} (\beta_{\mathcal{E}} \otimes \iota)(\beta_{\mathcal{E}}(w)e_i)x$$

$$= (\iota \otimes \Delta)\alpha_{\mathcal{I}}(v) \underset{\pi \otimes \iota \otimes \iota}{\otimes} (\iota \otimes \Delta)\beta_{\mathcal{E}}(w)x .$$

On the other hand, we have

$$(\mathcal{V} \underset{\iota \otimes \Delta}{\otimes} 1)\big((v \otimes w) \underset{\pi}{\underset{(\beta_B \otimes \iota)\beta_B}{\otimes}} x\big) = \lim_i (\mathcal{V} \underset{\iota \otimes \Delta}{\otimes} 1)\big(((v \otimes w) \underset{\pi}{\underset{\beta_B}{\otimes}} e_i) \underset{\iota \otimes \Delta}{\otimes} x\big) = \lim_i (\alpha_{\mathcal{I}}(v) \underset{\pi \otimes \iota}{\otimes} \beta_{\mathcal{E}}(w)e_i) \underset{\iota \otimes \Delta}{\otimes} x .$$

We identify

$$\big((\mathcal{I} \otimes M) \underset{\pi \otimes \iota}{\otimes} (\mathcal{E} \otimes \mathcal{K})\big) \underset{\iota \otimes \Delta}{\otimes} (B \otimes \mathcal{K} \otimes \mathcal{K}) \simeq (\mathcal{I} \otimes M \otimes M) \underset{\pi \otimes \iota \otimes \iota}{\otimes} (B \otimes \mathcal{K} \otimes \mathcal{K})$$

which is given by

$$(v \underset{\pi \otimes \iota}{\otimes} w) \underset{\iota \otimes \Delta}{\otimes} x \mapsto (\iota \otimes \Delta)(v) \underset{\pi \otimes \iota \otimes \iota}{\otimes} (\iota \otimes \Delta)(w)x .$$

Hence, we conclude that

$$(\mathcal{V} \underset{\iota \otimes \Delta}{\otimes} 1)\big((v \otimes w) \underset{\pi}{\underset{(\beta_B \otimes \iota)\beta_B}{\otimes}} x\big) = \lim_i (\iota \otimes \Delta)\alpha_{\mathcal{I}}(v) \underset{\pi \otimes \iota \otimes \iota}{\otimes} (\iota \otimes \Delta)(\beta_{\mathcal{E}}(w)e_i)x = (\iota \otimes \Delta)\alpha_{\mathcal{I}}(v) \underset{\pi \otimes \iota \otimes \iota}{\otimes} (\iota \otimes \Delta)(\beta_{\mathcal{E}}(w))x .$$

This proves our claim. From the lemma following this proposition, we get that $\mathcal{V} \in \mathcal{L}((\mathcal{I} \otimes \mathcal{E}) \underset{\beta_B}{\underset{\pi}{\otimes}} A, (\mathcal{I} \otimes \mathcal{E}) \underset{\pi}{\otimes} A)$ and that we get the desired coaction $\gamma_{\mathcal{F}}$ on $\mathcal{F} = \mathcal{I} \underset{\pi}{\otimes} \mathcal{E}$ as stated in the proposition. $\square$

Lemma 12.14. *Let $\mathcal{E}$ be a C^*-B-module and let $\beta_B : B \to \mathcal{M}(B \otimes A)$ be a coaction of a l.c. quantum group (A, Δ) on B. Suppose that $\mathcal{V} \in \mathcal{L}\big(\mathcal{E} \underset{\beta_B}{\otimes} (B \otimes \mathcal{K}(H)), \mathcal{E} \otimes \mathcal{K}(H)\big)$ is a unitary satisfying*

$$(\mathcal{V} \underset{\mathbb{C}}{\otimes} \iota)(\mathcal{V} \underset{\beta_B \otimes \iota}{\otimes} 1) = \mathcal{V} \underset{\iota \otimes \Delta}{\otimes} 1 .$$

Here, we denote $\Delta : \mathcal{K}(H) \to \mathcal{M}(\mathcal{K}(H) \otimes \mathcal{K}(H)) : \Delta(k) = W^*(1 \otimes k)W$ and we refer to (12.2) for the precise meaning of the formula satisfied by $\mathcal{V}$.

Then, $\mathcal{V} \in \mathcal{L}\big(\mathcal{E} \underset{\beta_B}{\otimes} (B \otimes A), \mathcal{E} \otimes A\big)$ and hence, there exists a unique coaction $\beta_{\mathcal{E}}$ of (A, Δ) on $\mathcal{E}$ compatible with β_B and satisfying

$$\beta_B(v)x = \mathcal{V}(v \underset{\beta_B}{\otimes} x) \quad \text{for all } v \in \mathcal{E}, x \in B \otimes A .$$

Proof. Since $W \in \mathcal{M}(A \otimes \mathcal{K}(H))$, we get that in fact $\Delta : \mathcal{K} \to \mathcal{M}(A \otimes \mathcal{K})$. From this, we conclude that

$$\mathcal{V} \underset{\iota \otimes \Delta}{\otimes} 1 \in \mathcal{L}(\mathcal{E} \underset{(\beta_B \otimes \iota)\beta_B}{\otimes} (B \otimes A \otimes \mathcal{K}), \mathcal{E} \otimes A \otimes \mathcal{K}) .$$

Further, $\beta_B : B \to \mathcal{M}(B \otimes A)$, from which we get that

$$\mathcal{V} \underset{\beta_B \otimes \iota}{\otimes} 1 \in \mathcal{L}(\mathcal{E} \underset{(\beta_B \otimes \iota)\beta_B}{\otimes} (B \otimes A \otimes \mathcal{K}), (\mathcal{E} \underset{\beta_B}{\otimes} (B \otimes A)) \otimes \mathcal{K}) .$$

But then, the formula satisfied by $\mathcal{V}$ guarantees that

$$\mathcal{V} \underset{\mathbb{C}}{\otimes} \iota \in \mathcal{L}((\mathcal{E} \underset{\beta_B}{\otimes} (B \otimes A)) \otimes \mathcal{K}, \mathcal{E} \otimes A \otimes \mathcal{K}) .$$

Hence, we find that $\mathcal{V} \in \mathcal{L}\big(\mathcal{E} \underset{\beta_B}{\otimes} (B \otimes A), \mathcal{E} \otimes A\big)$ $\square$

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